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SCATTERING BY RANDOMLY VARYING MEDIA WITH APPLICATION TO RADAR DETECTION AND COMMUNICATIONS

JANUARY 1968

Space and Missile Systems Organization
Air Force Systems Command
United States Air Force
Los Angeles, California 90045

Project No. 5218, Task No. 10

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NOTICES

ABSTRACT

This report is primarily concerned with the study of electromagnetic scattering by random scatterers. Potential applications to radar detection and communication problems are stressed. In radar detection problems, it is often necessary to detect a target echo in the presence of other unwanted echoes (clutter). In order that a radar receiver can be designed to operate effectively in the presence of clutter interference, it is necessary to develop a suitable theoretical model of the clutter which can be used for the design and evaluation of detection schemes. In radio communication problems, it is desirable that signals can be communicated between two non-line-of-sight points by means of electromagnetic scattering from a medium which occupies the region of space illuminated by the two antenna beams. Detailed knowledge of the scattering characteristics of the medium will enable one to select proper signal parameters for transmittal of information and optimum processing schemes.

A cloud of dipoles (chaff), dispensed in a proper region in space to act as radar reflectors, can be described as an assembly of random scatterers. When chaff dipoles are dispensed from a moving craft in space, they will in general move relative

to one another. In addition, each dipole will have a tumbling motion due to effects of injection forces, body instability, and other aerodynamical properties. Since the location and motion of the individual scatterers are unknown, the scattering problems are best treated statistically.

The bistatic radar reflecting characteristics of a randomly tumbling dipole are investigated. An expression for the scattered voltage is derived by application of the Lorentz reciprocity theorem. The correlation properties of the received signal are examined. Some statistical assumptions are made in order to obtain readily usable results.

A theoretical model is developed for the radar echo from a random collection of moving dipole scatterers. The analysis of the model takes into account some effects of scatterer rotation which have been neglected in previous work. The fluctuating characteristics of clutter echoes are also determined. The theory and some experimental results in the literature are shown to be in relatively good agreement.

The properties of random scatter communication channels are also investigated. The constitutive parameters of the scattering medium are assumed to be varying randomly with space and time. The effects of antenna gain are included in the derivation of the channel function in order to take explicit account

of the fact that scatterers may flow in and out of the volume illuminated by the two antenna beams. Arbitrary polarization is assumed for both the transmitting and receiving antennas. Specific results are obtained for dipole and plasma scattering.

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Chapter 1

INTRODUCTION

1.1 The Problem and Objectives of this Study

In radar systems it is often necessary to detect a target echo in the presence of other unwanted echoes or clutter. Clutter echoes include signals reflected from chaff, surface of the ground and sea, vegetation, etc. The ability of a radar to detect a target echo is limited by the presence of clutter. In order to develop a radar receiver that will operate effectively in the presence of clutter interference, it is desirable to have a theoretical model of the clutter which can be used for the design and evaluation of detection schemes. If the clutter waveform were precisely known, it would be possible, in principle, to synthesize a filter which optimizes the signal-to-noise ratio at the output of the receiver.

The clutter model most often assumed for theoretical analyses is the "point scatterer" model. Rotational motion of the scatterers is neglected. The scatterers are treated as point targets with variable cross section.

Chaff is a form of countermeasure used against radar. A chaff cloud usually consists of a large number of dipoles resembling an assembly of random scatterers. When chaff dipoles are dispensed from a moving craft in space, the dipoles experience certain forces which may cause them to rotate or tumble. In some applications the dipoles may be constrained to rotate about a preferred axis. Thus, ignoring the effects of scatterer rotation will result in an incorrect representation of the model. Ground clutter exhibits echoing characteristics similar to those of an assembly of random scatterers. Rotational motion of the scatterers, or equivalent movements of branches, leaves, grass, etc., under the effects of wind force, can become a major contributing factor to the observed fluctuations of echo intensity.

Herein, a new theoretical model for the radar echo from such collections of random scatterers is developed. The scatterers are treated as randomly oriented dipoles instead of points with variable cross sections. This assumption permits the determination of scatterer rotation.

In many radio communication problems signals must be communicated between two non-line-of-sight points by means of electromagnetic scattering from a medium which occupies the region

of space illuminated by the two antenna beams. The scattering medium may be continuous such as the inhomogeneous atmosphere or may consist of a random collection of metallic scatterers such as chaff.

The scattering medium in general varies with both position and time. In the analyses of random scattering most investigators have ignored the time dependence and have been concerned only with the calculation of the average cross section of the scattering medium. The average cross section is useful for estimating the signal-to-noise ratio at the receiver; however, it does not provide adequate information regarding the fluctuating characteristics of the scattered waveform. In the present work, we study scatter channels in which the scattering medium is randomly varying. General time dependence is assumed. The statistical nature of such channels are carefully examined.

1.2 Summary of Previous Work

Clutter echoes are often regarded as signals reflected from an assembly of independent random scatterers. A cloud of dipoles (chaff), dispensed randomly in space, is an example of such a set. Because of the random location and motion of the individual scatterers, clutter echoes are best treated statistically. Siegert [26],

Kerr [13], and Lawson and Uhlenbeck [27], developed the first probability distribution functions for the magnitude of the returned echo from an infinitely dense chaff cloud. Kelly and Lerner [24] developed a more extensive theoretical chaff model. They showed that the returned echo from the chaff cloud could be regarded as a random process and treated the scatterers as points with variable cross sections.

Kerr [13] and Lawson and Uhlenbeck [27] presented some experimental results on the measurements of chaff models and ground clutter. Some anomalous characteristics were observed in the measured correlation functions. Kerr suggested that the cause could have been due to the effects of scatterer rotation, but he made no theoretical analysis to clarify this phenomenon.

The possibility of using a cloud of chaff dipoles for establishing short time communications between non-line-of-sight points was considered by Blom [8] and Hessemer [9]. The recent West Ford Project [11] extended this basic concept and deployed an orbiting dipole belt. This belt established the first world-wide scatter communication channel.

Some of the statistical properties of random scatter channels were presented recently by Kelly [14]. Kelly's analyses include both

the continuous and discrete scatterer channels. He took into account quite generally the varying character of the constitutive parameters of the medium with position and time.

1.3 Summary of Results

Chapter 2 presents an analysis of the reflecting characteristics of a rotating dipole. The dipole is assumed to be rotating about an arbitrary axis perpendicular to its length. Arbitrary polarization is assumed for both the transmitting and receiving antennas. The Lorentz reciprocity theorem is applied to derive an expression for the bistatic scattered voltage. It is shown that the received signal is in general modulated in both amplitude and phase. The correlation properties of the received signal are examined. Some statistical assumptions are made in order to obtain tractable results.

A theoretical model for the radar echo from a random collection of dipole scatterers is formulated in Chapter 3. The analysis of the model takes into account the effects of scatterer rotation. An expression for the correlation function of the returned echo is derived in terms of the characteristics of the transmitted waveform, polarization, and the statistical properties of the scatterers. Under the assumption that the cloud density is slowly varying, the process becomes stationary. The probability density of the returned

waveform is shown to approach a Gaussian probability density as the rate of echo return becomes large. The theory can be applied to the study of a wide class of clutter signals which exhibit similar echoing characteristics. For example, targets making up ground clutter such as trees, grass, etc., will move and/or oscillate with the wind. The individual blades or stems scatter like dipoles. The fluctuating characteristics of clutter echoes are also determined. Comparison between the theory and some experimental results in the literature are shown to be in relatively good agreement.

Chapter 4 describes a communication channel which is provided by means of electromagnetic scattering from a collection of scatterers randomly distributed in a region of space. The collection of scatterers behaves like a dispersive medium, and the number density varies with both position and time. The channel response function is described in terms of the scattering properties of the medium. The mean and the covariance of the channel output are derived under the assumption of Poisson statistics. The gain functions of the transmitting and receiving antennas are included in the analysis in order to take explicit account of the fact that scatterers may flow in and out of the volume illuminated by the two antenna beams. The effects of scatterer rotation are also considered. The case of dipole scattering is given as an example.

The properties of a dispersive continuous channel are considered in Chapter 5. The scattered fields are derived in terms of the electric and magnetic susceptibility functions of medium. As in the discrete scatterer case, the effect of antenna gain is also taken into account in deriving the channel function. Since there are many similarities in the formalism between the two types of scattering, most of the results obtained for the discrete scatterer case are valid for the continuous case.

Chapter 2

BISTATIC RADAR REFLECTIONS FROM A RANDOMLY ROTATING DIPOLE

2.1 Introduction

When a plane electromagnetic wave impinges upon a target in space, some of the incident energy will be scattered in the direction of the radar receiver. The characteristics of the scattered signal depend upon (1) the size, shape, and orientation of the target and (2) the frequency and state of polarization of the incident wave.

The problem of electromagnetic scattering from cylindrical wires has been considered by a number of investigators [1] - [7]. A bundle of metallic wires or strips (known as chaff), cut to resonate at a certain wavelength, may be dispensed in a proper region in space to act as effective radar reflectors. These reflectors can perform two different functions:

(1) To present certain characteristic echoes to a radar for purposes of identification, deception, or confusion [3].

(2) To provide a channel for beyond line-of-sight communications [8] - [11].

A cloud of dipoles can be represented by an assembly of random scatterers. When chaff dipoles are dropped into space, they will in general move relative to one another. Each dipole will have a slightly varying position and velocity, thus giving rise to a Doppler spread in the scattered signal. In addition, the dipoles will have a tumbling motion due to effects of body instability and other aerodynamical properties. Tumbling motion of the dipoles can cause further frequency spreading in the received waveform. In many practical situations, tumbling speeds were not thought to be very rapid [11] - [12] and considered to give only second order effects; while in other situations, the effects of scatterer tumbling (or rotation) could be observed and found to have significant contributions upon the received waveform [13]. It will be shown in Chapter 3 that, in general, the effects due to scatterer rotation cannot be neglected if the Doppler spread arising from the velocity fluctuations of the scatterers is small.

In order that a radar receiver can be designed to operate in an optimum fashion, an adequate knowledge of the amplitude and phase characteristics of the scattered waveform will be required. In the analyses of random scattering, most investigators ignore the effects of scatterer rotation and assume a "point scatterer" model, using the information on the average cross section of the individual

scatterers. The "point scatterer" model is valid for rotationally symmetric scatterers. However, such a representation is not adequate for scatterers of other shapes such as dipoles.

This chapter presents an analysis of the reflecting characteristics of a tumbling dipole. It is assumed that the tumbling motion of the dipole can be described by a rotation about an axis perpendicular to its length. An expression for the bistatic scattered voltage is derived as a function of the transmitter and receiver polarizations and the orientation of the rotation axis of the dipole. The statistical properties of the signal reflected from a collection of randomly rotating chaff dipoles are discussed by means of a mathematical model. The correlation function of the received signal envelope is derived in terms of the characteristics of the transmitted waveform, polarization, and the statistical properties of the scatterers.

2.2 Characteristics of the Scattered Signal

An expression for the scattered signal can be derived by application of the Lorentz reciprocity theorem (Appendix I). The radar system consists of a transmitter which illuminates the target and a receiver at which we evaluate the scattered signal. The total voltage received is the sum of the voltages induced by the transmitter field and the field scattered by the target. In the following derivation,

we shall ignore the coupling effects between the transmitter and receiver, but allow for variable polarizations. The received voltage is then given by

$$\mathcal{V} = \frac{1}{2 I_R} \int_{V_s} \bar{E}_R \cdot \bar{J} \, dV \tag{2.1}$$

where I_R is the feed-point current which would have to be supplied to the receiving antenna, if it were used as a transmitting antenna, to create the field $\bar{E}_R$ in the absence of the target, and $\bar{J}$ is the current density associated with the dipole. If the dipole is thin and if the field can be considered uniform throughout the small volume V_s containing the dipole, the integral on the right side of (2.1) can be expressed in terms of the induced moment on the dipole. Thus,

$$\int_{V_s} \bar{E}_R \cdot \bar{J} \, dV = i\omega_c \bar{p} \cdot \bar{E}_R \tag{2.2}$$

The induced dipole moment can be written as

$$\bar{p} = \epsilon_o \alpha_e (\hat{d} \cdot \bar{E}_T) \hat{d} \tag{2.3}$$

where $\bar{E}_T$ is the transmitted electric field, α_e is the polarizability of the dipole[1], and $\hat{d}$ is a unit vector parallel to the dipole axis. The

[1] Here the polarizability is treated as a scalar; more generally, it is a tensor [14].

field quantities $\bar{E}_T$ and $\bar{E}_R$ can be written as $\bar{E}_T = \tilde{E}_{OT}\hat{e}_T$ and $\bar{E}_R = \tilde{E}_{OR}\hat{e}_R$, where $\hat{e}_T$ and $\hat{e}_R$ are the polarization vectors of the transmitter and receiver, respectively, and the symbol ~ is used to indicate a complex quantity. Equation (2.1) becomes

$$\tilde{\gamma} = \frac{i\omega_o\epsilon_o\alpha_e}{2\,I_R}\tilde{E}_{OT}\tilde{E}_{OR}(\hat{d}\cdot\hat{e}_T)(\hat{d}\cdot\hat{e}_R) \qquad (2.4)$$

If the same antenna is used for transmitting and receiving (monostatic case), we have $\hat{e}_T = \hat{e}_R = \hat{e}$ and $\tilde{E}_{OT} = \tilde{E}_{OR} = \tilde{E}_O$, (2.4) reduces to

$$\tilde{a} = \frac{i\omega_o\epsilon_o\alpha_e}{2\,I_o}\tilde{E}_o^{\,2}(\hat{d}\cdot\hat{e})^2 \qquad (2.5)$$

The bistatic scattering geometry is shown in Fig. 1. The transmitter and receiver are separated by the angle β. The incident wave is assumed to be propagating along the positive z-axis of the transmitter (unprimed) coordinate system, and the scattered wave is observed in the direction of the negative z'-axis of the receiver (primed) coordinate system. In order to compute the bistatic scattered voltage, it is necessary to express the dipole orientation and the polarization vectors in one common coordinate system. For simplicity, we choose the bistatic angle β such that it represents a

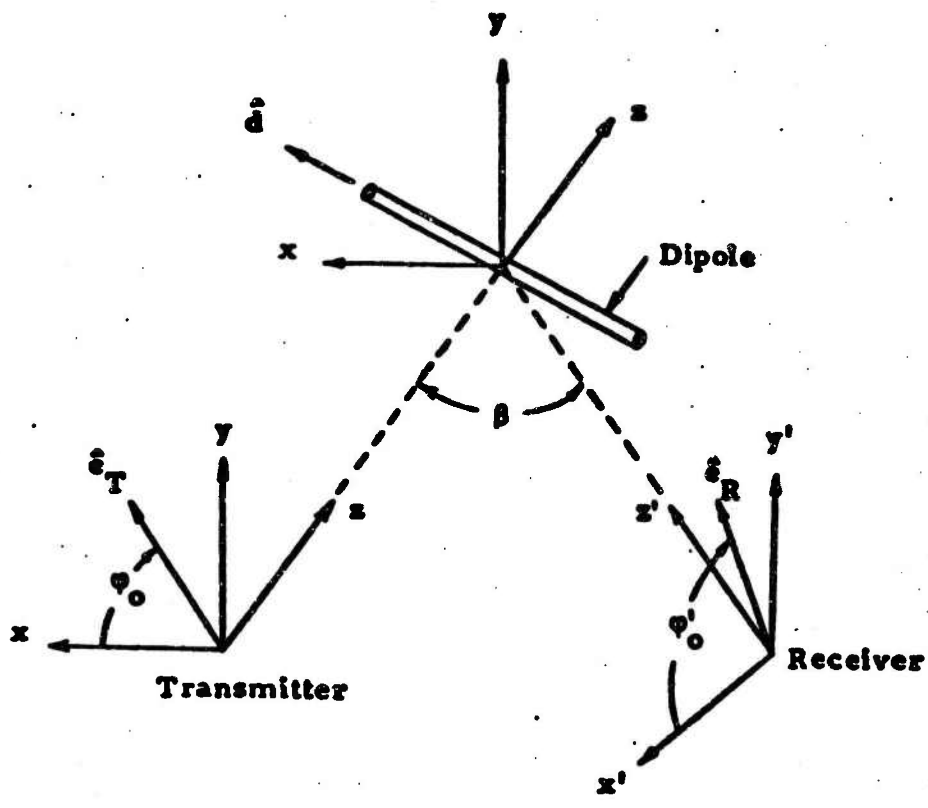

Fig. 3 - Bistatic Scattering Geometry

rotation in the xz-plane. Hence, the unit vectors $(\hat{a}'_x, \hat{a}'_y, \hat{a}'_z)$ in the receiver coordinate system are related to the unit vectors $(\hat{a}_x, \hat{a}_y, \hat{a}_z)$ in the transmitter coordinate system by the rotational transformation,

$$\begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \tag{2.6}$$

Figure 2 shows the coordinate system for the rotating dipole. For the present purpose, the distance between the dipole and the radar is assumed to be fixed. The dipole is rotating about an arbitrary axis s defined by the angles ξ and η. Let (u, v, s) be a set of orthogonal axes, then u and v lie in the plane of rotation of the dipole. If ω_r is the angular rotation frequency of the dipole and if α is its initial position with respect to the u-axis, then the instantaneous position of the unit vector $\hat{d}$ may be represented by

$$\hat{d} = \hat{a}_u \cos\psi + \hat{a}_v \sin\psi \tag{2.7}$$

where $\psi = \omega_r t + \alpha$ and $\hat{a}_u$ and $\hat{a}_v$ are unit vectors along the u and v axes, respectively. For convenience, the u-axis is chosen to be coincident with the intersection of the xy-plane and the plane of rotation of the dipole; then, by means of a transformation between the (x, y, z) and (u, v, s) coordinate systems [15], we obtain

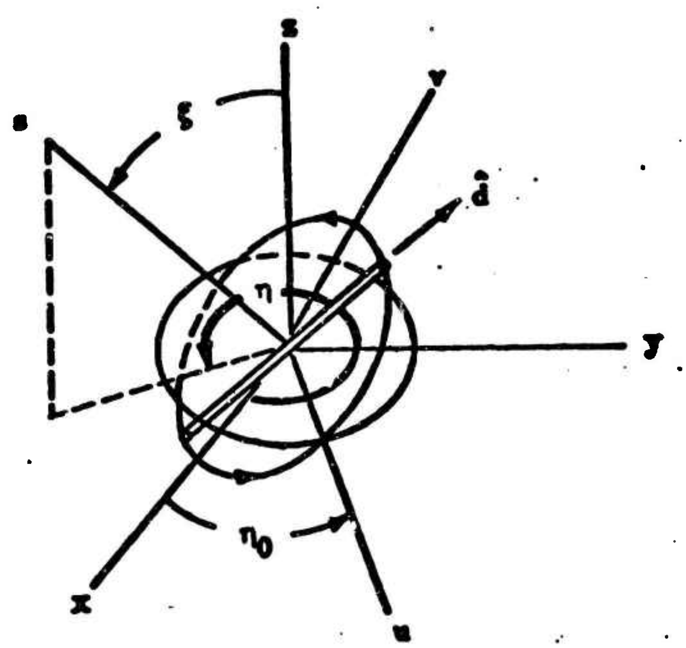

Fig. 2 - Coordinate System for Rotating Dipole

$$\hat{d} = [-\hat{a}_x(\sin\eta\cos\psi + \cos\xi\cos\eta\sin\psi)$$
$$+\hat{a}_y(\cos\eta\cos\psi - \cos\xi\sin\eta\sin\psi)$$
$$+\hat{a}_z(\sin\xi\sin\psi)] \tag{2.8}$$

where $(\hat{a}_x, \hat{a}_y, \hat{a}_z)$ are unit vectors along the (x, y, z)-axes, respectively.

The polarization vector of an arbitrary polarized plane wave can be written as the sum of a right-handed circularly polarized wave and a left-handed circularly polarized wave. Then, for the transmitter polarization, we write

$$\hat{e}_T = \tilde{a}_r \hat{\mathcal{R}} + \tilde{a}_\ell \hat{\mathcal{L}} \tag{2.9}$$

where

$$\hat{\mathcal{R}} = \frac{1}{\sqrt{2}}(\hat{a}_x - i\,\hat{a}_y)$$

$$\hat{\mathcal{L}} = \frac{1}{\sqrt{2}}(\hat{a}_x + i\,\hat{a}_y)$$

and $\tilde{a}_r$ and $\tilde{a}_\ell$ are complex constants satisfying the relation,

$$|\tilde{a}_r|^2 + |\tilde{a}_\ell|^2 = 1$$

Thus the state of polarization of the transmitted plane wave is completely determined once the complex quantities $\tilde{a}_r$ and $\tilde{a}_\ell$ are known;

e.g., for circular polarization, we have either ($\tilde{a}_r = 1$, $\tilde{a}_\ell = 0$) or ($\tilde{a}_r = 0$, $\tilde{a}_\ell = 1$) and for linear polarization, we have $\tilde{a}_r = \tilde{a}_\ell^* = (1/\sqrt{2})\, e^{i\varphi_o}$ where φ_o is the angle between the electric field vector and the x-axis as shown in Fig. 1. Similarly, the receiver polarization can be expressed in terms of the primed coordinate system. Thus,

$$\hat{e}_R = \tilde{a}'_r \hat{\mathcal{R}}' + \tilde{a}'_\ell \hat{\mathcal{L}}' \tag{2.10}$$

where

$$\hat{\mathcal{R}}' = \frac{1}{\sqrt{2}}(\hat{a}'_x - i\,\hat{a}'_y)$$

$$\hat{\mathcal{L}}' = \frac{1}{\sqrt{2}}(\hat{a}'_x + i\,\hat{a}'_y)$$

and

$$|\tilde{a}'_r|^2 + |\tilde{a}'_\ell|^2 = 1$$

Substituting of (2.8), (2.9), and (2.10) into (2.4) together with the aid of (2.6) yields, after some algebraic manipulations,

$$\tilde{a} = \tilde{a}_o\left[\tilde{C} + \tilde{U}\, e^{+i2\psi} + \tilde{L}\, e^{-i2\psi}\right] \tag{2.11}$$

where

$$\tilde{a}_o = \frac{i\omega_o \epsilon_o a_e}{2 I_R}\, \tilde{E}_{OT} \tilde{E}_{OR} \tag{2.12}$$

$$\tilde{C} = -\frac{1}{2}(\tilde{A}\tilde{A}' - \tilde{B}\tilde{B}') \tag{2.13}$$

$$\tilde{U} = -\frac{1}{4}(\tilde{A} + \tilde{B})(\tilde{A}' + \tilde{B}') \tag{2.14}$$

$$\tilde{L} = -\frac{1}{4}(\tilde{A} - \tilde{B})(\tilde{A}' - \tilde{B}') \tag{2.15}$$

$$\tilde{A} = \frac{1}{\sqrt{2}}(\tilde{a}_\ell e^{+i\eta} - \tilde{a}_r e^{-i\eta}) \tag{2.16}$$

$$\tilde{B} = \frac{1}{\sqrt{2}}(\tilde{a}_\ell e^{+i\eta} + \tilde{a}_r e^{-i\eta}) \cos \xi \tag{2.17}$$

$$\tilde{A}' = \frac{1}{\sqrt{2}}\Big[\Big(\tilde{a}'_\ell \cos^2\frac{\beta}{2} - \tilde{a}'_r \sin^2\frac{\beta}{2}\Big)e^{+i\eta} + \Big(\tilde{a}'_\ell \sin^2\frac{\beta}{2} - \tilde{a}'_r \cos^2\frac{\beta}{2}\Big)e^{-i\eta}\Big] \tag{2.18}$$

$$\tilde{B}' = \frac{1}{\sqrt{2}}\Big\{\Big[\Big(\tilde{a}'_\ell \cos^2\frac{\beta}{2} - \tilde{a}'_r \sin^2\frac{\beta}{2}\Big)e^{+i\eta} - \Big(\tilde{a}'_\ell \sin^2\frac{\beta}{2} - \tilde{a}'_r \cos^2\frac{\beta}{2}\Big)e^{-i\eta}\Big]\cos\xi - \Big[\Big(\tilde{a}'_\ell + \tilde{a}'_r\Big)\sin\beta\Big]\sin\xi\Big\} \tag{2.19}$$

For the case of monostatic radar ($\beta = 0$), i.e., the same antenna is used for transmitting and receiving, we have $\tilde{E}_{OR} = \tilde{E}_{OT}$, $\tilde{A}' = \tilde{A}$, and $\tilde{B}' = \tilde{B}$. The quantity $\tilde{a}_o$ may be interpreted as the maximum scattered voltage with both the transmitter and receiver polarization

vectors parallel to the dipole axis.

The first term in (2.11) is independent of ψ and hence its amplitude does not vary with time. The received signal consists of three spectral components, namely a carrier component plus the upper and lower sideband components separated from the carrier by twice the rotational frequency of the dipole. Furthermore, the amplitudes of the upper and lower sideband components are in general not equal, depending on the radar polarization and the orientation of the rotation axis. Thus, both amplitude and phase modulation of the scattered signal can result.

Equation (2.11) is a general expression for the bistatic scattered voltage with arbitrary transmitter and receiver polarizations. As a special example, consider the case of backscattering of a circularly polarized wave. Assume that both the transmitting and receiving antennas are polarized in the same sense and that the rotation axis is the positive z-axis (clockwise rotation about the line of sight). Using Eqs. (2.11) to (2.19) and setting $\psi = \omega_r t + \alpha$, we find

$$\tilde{a} = -\frac{1}{2}\tilde{a}_o e^{-i2(\omega_r t + \alpha)} \qquad (2.20)$$

for the right-handed circular polarization ($\tilde{a}_r = \tilde{a}_r' = 1$ and $\tilde{a}_\ell = \tilde{a}_\ell' = 0$), and

$$\tilde{a} = -\frac{1}{2}\tilde{a}_o e^{+i2(\omega_r t + \alpha)} \tag{2.21}$$

for left-handed circular polarization ($\tilde{a}_\ell = \tilde{a}'_\ell = 1$ and $\tilde{a}_r = \tilde{a}'_r = 0$). Since the scattered voltage is a real function of time, we multiply (2.20) and (2.21) by $e^{+i\omega_o t}$ and evaluate the real part. Hence, we obtain

$$a = \begin{cases} \frac{1}{2}\,|\tilde{a}_o| \cos\left[(\omega_o - 2\omega_r)t + \gamma_R\right] \text{-----RHCP} \\ \frac{1}{2}\,|\tilde{a}_o| \cos\left[(\omega_o + 2\omega_r)t + \gamma_L\right] \text{-----LHCP} \end{cases}$$

where γ_R and γ_L are constants, ω_o is the angular carrier frequency of the transmitted wave, and ω_r is the rotation frequency of the dipole. Thus, the original transmitter frequency is shifted. The frequency shift is either downward or upward depending on whether the polarization is right-handed or left-handed, respectively. More precisely, the frequency of the returned signal is shifted down if the polarization vector and the dipole are rotating in the same direction and is shifted up if they are rotating in opposite directions. This result has also been shown by Mikulski [16] and Montgomery [17] by means of periodic scattering matrices.

2.3 The Bistatic Scattering Cross Section

An expression for the bistatic cross section can be derived by considering the power flow in the transmitted and the received waves. Thus, the average power received is

$$P_s = \frac{1}{2Z_o} |\tilde{a}|^2 \tag{2.22}$$

where Z_o is the characteristic impedance of the transmission line. Substituting (2.4) into (2.22), we obtain

$$P_s = \frac{\omega_o^2 \epsilon_o^2 \alpha_e^2}{4 I_R^2} |\tilde{E}_{OT}|^2 |\tilde{E}_{OR}|^2 |(\hat{d} \cdot \hat{e}_T)(\hat{d} \cdot \hat{e}_R)|^2 \tag{2.23}$$

Let P_T be the average power supplied to the transmitting antenna to create the field $\tilde{E}_{OT} \hat{e}_T$ in free space and let G_T be the gain function, then the time average magnitude of the Poynting vector at a distance r_T from the transmitter is

$$|\bar{S}_T| = \frac{G_T P_T}{4\pi r_T^2} = \frac{1}{2}\sqrt{\frac{\epsilon_o}{\mu_o}} |\tilde{E}_{OT}|^2 \tag{2.24}$$

Similar definitions can be applied to the receiving antenna. Thus,

$$|\bar{S}_R| = \frac{G_R P_R}{4\pi r_R^2} = \frac{1}{2}\sqrt{\frac{\epsilon_o}{\mu_o}} |\tilde{E}_{OR}|^2 \tag{2.25}$$

where $P_R = \frac{1}{2} Z_o I_R^2$ is the power which must be supplied to the receiving antenna in order to create the field $\tilde{E}_{OR} \hat{e}_R$ in free space. Combining (2.23), (2.24), and (2.25), we obtain

$$P_s = \frac{\pi^2 \alpha_e^2}{\lambda^2} \frac{G_T G_R P_T}{(4\pi)^2 r_T^2 r_R^2} \left|(\hat{d} \cdot \hat{e}_T)(\hat{d} \cdot \hat{e}_R)\right|^2 \tag{2.26}$$

If Eq. (2.26) is compared with the bistatic radar equation,

$$P_s = \frac{G_T G_R P_T \lambda^2}{(4\pi)^3 r_T^2 r_R^2} \sigma \tag{2.27}$$

we find,

$$\sigma = \frac{1}{4\pi} k_o^4 \alpha_e^2 \left|(\hat{d} \cdot \hat{e}_T)(\hat{d} \cdot \hat{e}_R)\right|^2 \tag{2.28}$$

where σ is the bistatic cross section and $k_o = 2\pi/\lambda$ is the free space wave number. The quantity $k_o^4 \alpha_e^2 / 4\pi$ is just the broadside cross section of the dipole when the incident electric field vector is parallel to the dipole axis [1]. For a thin dipole, the polarizability is approximately given by [18],

$$\alpha_e \cong \frac{\pi \ell^3}{6\left[\ln\left(\frac{2\ell}{a}\right) - \frac{7}{3}\right]}$$

where ℓ and a are the length and radius of the dipole, respectively.

This approximation is valid when $\frac{\ell}{a} \gg 1$. Using (2.11), (2.28) may be written as

$$\frac{\sigma}{\sigma_o} = \left|\left(\tilde{C} + \tilde{U}\, e^{+i2\psi} + \tilde{L}\, e^{-i2\psi}\right)\right|^2 . \qquad (2.29)$$

where

$$\sigma_o = \frac{1}{4\pi} k_o^4 a_e^2 \qquad (2.30)$$

Equation (2.29) also applies to small resonant (tuned) dipoles and is approximately valid for half-wave resonant dipoles, since the difference between the patterns of a half-wave dipole and a short dipole is small. However, the quantity σ_o must be replaced appropriately; for example, the broadside cross section of a small resonant dipole is approximately given by [18],

$$\sigma_o \approx \frac{9}{4\pi} \lambda^2 = .716\, \lambda^2 \qquad (2.31)$$

and that of a half-wave resonant dipole is [5] - [6],

$$\sigma_o \approx .86\lambda^2 \qquad (2.32)$$

Of particular interest is the computation of the average bistatic cross section of a randomly rotating dipole. Assume that the initial orientation of the dipole axis in space is uniformly distributed and that every orientation of the rotation axis in space is

equally probable, then, from Eq. (2.29), we have

$$\langle\sigma\rangle = \frac{\sigma_o}{4\pi}\int_0^{2\pi}\int_0^{\pi}(C^2 + U^2 + L^2)\sin\xi\, d\xi\, d\eta$$

$$= \sigma_o \langle C^2 + U^2 + L^2\rangle \qquad (2.33)$$

where C, U and L are the magnitudes of $\tilde{C}$, $\tilde{U}$, and $\tilde{L}$, respectively, and the symbol < > is used to denote an ensemble average. If we denote the quantity $\langle C^2 + U^2 + L^2\rangle$ by $\langle b_o^2\rangle$, (2.33) can be written as

$$\langle\sigma\rangle = \sigma_o \langle b_o^2\rangle \qquad (2.34)$$

Using Equations (2.13) through (2.19), we find

$$\langle b_o^2\rangle = \frac{1}{15}\Big\{1 + (|\tilde{a}_\ell|^2 - |\tilde{a}_r|^2)(|\tilde{a}'_\ell|^2 - |\tilde{a}'_r|^2)\cos\beta \qquad (2.35)$$

$$+ \frac{1}{2}\left|[(\tilde{a}_\ell - \tilde{a}_r)(\tilde{a}'_\ell - \tilde{a}'_r) - (\tilde{a}_\ell + \tilde{a}_r)(\tilde{a}'_\ell + \tilde{a}'_r)\cos\beta]\right|^2\Big\}$$

Equation (2.35) is a general expression for the average relative bistatic cross section with arbitrary transmitter and receiver polarizations. Hence, if the radar system is equipped with polarization diversity capability, the polarization of the transmitter, receiver, or both can be varied to achieve either maximum or

minimum cross section as desired. For the case of backscattering, the average cross section becomes

$$\langle \sigma_b \rangle = \frac{\sigma_o}{15} \left\{ 1 + \left(|\tilde{a}_\ell|^2 - |\tilde{a}_r|^2 \right) \left(|\tilde{a}'_\ell|^2 - |\tilde{a}'_r|^2 \right) + 2 \left| \left(\tilde{a}_\ell \tilde{a}'_r + \tilde{a}'_\ell \tilde{a}_r \right) \right|^2 \right\} \tag{2.36}$$

Since the process is random, the measured cross section usually fluctuates about the average value. The probability density functions for the radar cross section of N randomly-oriented dipoles have been discussed by Borison [19], Fielding [20], and Rheinstein [21].

2.4 Signal Correlation Properties

The average cross section provides useful information for computing the expected signal to noise ratio at the receiver [11], [22]. However, knowledge of the average cross section alone is not sufficient for optimum radar receiver design since only very little information is provided regarding the fluctuating characteristics of the received waveform. Proper design of radar receivers requires a knowledge of some of the statistical properties of the scattered waveform. Specifically, we need a suitable model for the chaff cloud from which we can deduce various statistical properties of the received waveform. For design purposes, the most important

information may be considered to be contained in the received signal correlation function [23].

Consider a chaff cloud composed of a large number of dipoles which are moving about and reflecting energy independently of one another. In addition, each dipole is assumed to be continuously tumbling (rotating) about an axis perpendicular to its length. Assume that the transmitted signal is of the form

$$e_o(t) = Re[\tilde{s}(t)e^{i\omega_o t}] \qquad (2.37)$$

where

$$\tilde{s}(t) = |\tilde{s}(t)|e^{i\phi_o(t)} \qquad (2.38)$$

is the complex modulation envelope and ω_o is the angular carrier frequency. Let $\tilde{a}_k(t)$ be the reflection coefficient of the k^{th} scatterer, and r_k and r_k' be the distances from the k^{th} scatterer to the transmitter and receiver, respectively. The complex envelope of the received signal over an observation interval (-T, T) can be represented by [24], [25],

$$\tilde{z}(t) = \sum_{t_k \epsilon(-T, T)} \tilde{g}_k \tilde{a}_k(t-t_k + \frac{r_k}{c}) \tilde{s}(t-t_k)e^{i\omega_{dk} t} \qquad (2.39)$$

where the subscript k designates the k^{th} scatterer, $t_k = (r_k + r_k')/c$

is the time delay between the transmitter and receiver, ω_{dk} is the Doppler frequency shift, and $\tilde{g}_k$ is a complex number which may be a function of the positional coordinates, the antenna beam characteristics, and other initial parameters. From Eq. (2.11), the reflection coefficient can be written as

$$\tilde{a}_k(t) = \tilde{a}_{ok} \left\{ \tilde{C}(\xi_k, \eta_k) + \tilde{U}(\xi_k, \eta_k) \exp\left[+ i2(\omega_{rk} t + \alpha_k)\right] \right.$$

$$\left. + \tilde{L}(\xi_k, \eta_k) \exp\left[-i2(\omega_{rk} t + \alpha_k)\right] \right\} \qquad (2.40)$$

The following simplifying assumptions are made regarding the scatterers:

(1) The scatterers are statistically independent and identically distributed.

(2) The distribution of the scatterers obeys Poisson statistics, i.e., the joint probability for n reflections to occur in the time intervals $(t_1, t_1 + dt_1)$, $(t_2, t_2 + dt_2), \ldots\ldots, (t_n, t_n + dt_n)$ is given by

$$p(t_1, t_2, \ldots\ldots, t_n; n)\, dt_1 dt_2 \ldots\ldots dt_n$$

$$= \prod_{k=1}^{n} \frac{\nu(t_k)}{k} \exp\left[-\int_{-T}^{T} \nu(t)\, dt\right] dt_k \qquad (2.41)$$

where $\nu(t)$ is the rate of signal arrival at the receiver.

(3) The velocity of the scatterers and the magnitude of the scattered signal are weakly correlated within the observation interval $(-T, T)$.

(4) The orientation of the rotation axis is not greatly affected by the motion of the scatterers.

(5) The initial orientation angle α_k of the dipole axis is uniformly distributed in the interval $0 \le \alpha_k \le 2\pi$.

For simplicity, we assume $<z(t)> = 0$ and the correlation function is given by

$$R(t_1, t_2) = \frac{1}{2} <\tilde{z}^*(t_1)\tilde{z}(t_2)>$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \int_{-T}^{T} \cdots \int \prod_{k=1}^{n} \frac{\nu(t_k)}{k} \exp\left[-\int_{-T}^{T} \nu(t)\, dt\right] dt_k$$

$$\cdot \int \cdots \int_{w_1} p(\vec{g}, \vec{a}_o, \vec{\xi}, \vec{\eta}, \vec{\alpha}, \vec{\nu}_d, \vec{\nu}_r) d\vec{g} \cdots d\vec{\nu}_r [\tilde{z}^*(t_1)\tilde{z}(t_2)]$$

(2.42)

with

$$\int \cdots \int_{w_1} p(\vec{g}, \vec{a}_o, \ldots, \vec{\omega}_r)\, d\vec{g} \ldots d\vec{\omega}_r$$

$$= \int_{-\infty}^{+\infty} \cdots \int p(\vec{g})\, d\vec{g} \ \ldots \int_{-\infty}^{+\infty} \cdots \int p(\vec{\omega}_r)\, d\vec{\omega}_r$$

and

$$p(\vec{w})\, d\vec{w} = \prod_{i=1}^{n} p(w_i)\, dw_i$$

where $\vec{w}$ represents $\vec{g}, \vec{a}_o, \ldots,$ etc. Substituting (2.39) and (2.40) into (2.42), we have

$$R(t_1, t_2) = K\phi_{\omega_d}(\tau) M(\tau) \exp\left[-\int_{-T}^{T} \nu(t)\, dt\right]$$

$$\cdot \sum_{n=0}^{\infty} \int_{-T}^{T} \cdots \int \prod_{k=1}^{n} \frac{\nu(t_k)}{k}\, dt_k$$

$$\cdot \sum_{k=1}^{n} \tilde{s}^*(t_1 - t_k)\, \tilde{s}(t_2 - t_k)$$

$$= K\phi_{\omega_d}(\tau) M(\tau) \exp\left[-\int_{-T}^{T} \nu(t)\, dt\right]$$

$$\cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int_{-T}^{T} \nu(t)\, dt\right]^{n-1} n$$

$$\cdot \int_{-T}^{T} \nu(t')\, \tilde{s}^*(t_1 - t')\, \tilde{s}(t_2 - t')\, dt'$$

$$= K\phi_{\omega_d}(\tau) M(\tau) \int_{-T}^{T} \nu(t')\, \tilde{s}^*(t_1 - t')\, \tilde{s}(t_2 - t')\, dt' \qquad (2.43)$$

where

$$K = \frac{1}{2} <|\tilde{g}|^2> <|\tilde{a}_o|^2> \tag{2.44}$$

$$\tau = t_2 - t_1 \tag{2.45}$$

$$M(\tau) = [<C^2> + <U^2> \phi_{\omega_r}(2\tau) + <L^2> \phi_{\omega_r}(-2\tau)] \tag{2.46}$$

$$\phi_{\omega_d}(\tau) = \int_{-\infty}^{+\infty} p(\omega_d)\, e^{i\omega_d \tau}\, d\omega_d$$

$$= \text{characteristic function of } p(\omega_d) \tag{2.47}$$

$$\phi_{\omega_r}(\tau) = \int_{-\infty}^{+\infty} p(\omega_r)\, e^{i\omega_r \tau}\, d\omega_r$$

$$= \text{characteristic function of } p(\omega_r) \tag{2.48}$$

and C, U, and L are magnitudes of $\tilde{C}$, $\tilde{U}$, and $\tilde{L}$, respectively. Equation (2.43) expresses the time-varying correlation of the received waveform. However, if $\nu(t)$ varies only slowly with time and if the observation interval is long compared with signal duration, (2.43) reduces to

$$R(\tau) = K\nu_o \phi_{\omega_d}(\tau) M(\tau) \int_{-\infty}^{+\infty} \tilde{s}^*(u-\tau)\, \tilde{s}(u)\, du \tag{2.49}$$

where ν_o is the average rate of signal arrival at the receiver. Thus, the correlation function depends only on the time difference

(t_2-t_1) and the process becomes stationary. Furthermore, if the dipoles are not rotating ($\omega_r = 0$), (2.49) reduces to

$$R(\tau) = K\nu_o \phi_{\omega_d}(\tau) < b_o^2 > \int_{-\infty}^{+\infty} \tilde{s}^*(u-\tau)\, \tilde{s}(u)\, du \qquad (2.50)$$

where $< b_o^2 > = < C^2 + U^2 + L^2 >$ is the relative average cross section.

2.5 Evaluation of Scattered Power in Spectral Components

The computation of the received signal correlation function requires the values of $< C^2 >$, $< U^2 >$, and $< L^2 >$. These quantities represent the relative magnitudes of the average power contained in the three spectral components. Assume that all orientations of the rotation axis of the dipole are equally likely. Then, from Equations (2.13) through (2.19), we find

$$< C^2 > = \frac{1}{60} \Big\{ 1 + (|\tilde{a}_\ell|^2 - |\tilde{a}_r|^2)(|\tilde{a}'_\ell|^2 - |\tilde{a}'_r|^2) \cos\beta$$
$$+ \frac{7}{4} \left| [(\tilde{a}_\ell - \tilde{a}_r)(\tilde{a}'_\ell - \tilde{a}'_r) - (\tilde{a}_\ell + \tilde{a}_r)(\tilde{a}'_\ell + \tilde{a}'_r) \cos\beta] \right|^2 \Big\} \qquad (2.51)$$

$$< U^2 > = < L^2 > = < S^2 >$$
$$= \frac{1}{40} \Big\{ 1 + (|\tilde{a}_\ell|^2 - |\tilde{a}_r|^2)(|\tilde{a}'_\ell|^2 - |\tilde{a}'_r|^2) \cos\beta$$
$$+ \frac{1}{12} \left| [(\tilde{a}_\ell - \tilde{a}_r)(\tilde{a}'_\ell - \tilde{a}'_r) - (\tilde{a}_\ell + \tilde{a}_r)(\tilde{a}'_\ell + \tilde{a}'_r) \cos\beta] \right|^2 \Big\} \qquad (2.52)$$

The received signal depends upon the receiving polarization as well as the transmitting polarization. Thus, by properly choosing the values of $\tilde{a}_{\ell}$, $\tilde{a}_r$, $\tilde{a}'_{\ell}$, and $\tilde{a}'_r$ as in Equations (2.9) and (2.10), the scattered power for any polarization combination can be obtained. In many practical situations, either linear or circular polarization is employed. For the case of linear polarization, we have

$$<C^2> = \frac{1}{60}\left[1 + 7(\sin\varphi_o \sin\varphi'_o + \cos\varphi_o \cos\varphi'_o \cos\beta)^2\right] \quad (2.53)$$

$$<S^2> = \frac{1}{40}\left[1 + \frac{1}{3}(\sin\varphi_o \sin\varphi'_o + \cos\varphi_o \cos\varphi'_o \cos\beta)^2\right] \quad (2.54)$$

$$<b_o^2> = \frac{1}{15}\left[1 + 2(\sin\varphi_o \sin\varphi'_o + \cos\varphi_o \cos\varphi'_o \cos\beta)^2\right] \quad (2.55)$$

where φ_o and φ'_o are defined in Fig. 1. For this case, it is noted that minimum received power is observed when

$$(\sin\varphi_o \sin\varphi'_o + \cos\varphi_o \cos\varphi'_o \cos\beta) = 0 \quad (2.56)$$

and maximum received power is observed when

$$(\sin\varphi_o \sin\varphi'_o + \cos\varphi_o \cos\varphi'_o \cos\beta) = \pm 1 \quad (2.57)$$

Equation (2.56) can be satisfied for many combinations of φ_o, φ'_o, and β, while Eq. (2.57) can be satisfied only for limited values of φ_o, φ'_o, and β; i.e., (1) $\beta = 0$ and the transmitter and receiver polarizations are parallel to each other and (2) $\beta \neq 0$ and both the transmitter and receiver polarizations are perpendicular to the

plane of incidence ($\varphi_o = \pm 90^\circ$ and $\varphi'_o = \pm 90^\circ$). Similar conclusions have also been observed by Borison [6].

For the case of circular polarization, we find

$$<C^2> = \frac{1}{60}\left[(1 \pm \cos\beta) + \frac{7}{4}(1 \mp \cos\beta)^2\right] \qquad (2.58)$$

$$<S^2> = \frac{1}{40}\left[(1 \pm \cos\beta) + \frac{1}{12}(1 \mp \cos\beta)^2\right] \qquad (2.59)$$

$$<b_o^2> = \frac{1}{10}\left[1 + \frac{1}{3}\cos^2\beta\right] \qquad (2.60)$$

where the upper sign corresponds to the case when both the transmitter and receiver polarizations are circularly polarized in the same sense and the lower sign corresponds to the case of the opposite sense. The received power is a function of the bistatic angle β and the cross section is independent of the sense of polarization.

2.6 Conclusions

The signal scattered from a rotating dipole consists of three spectral components - a carrier component, an upper sideband component, and a lower sideband oomponent. The sideband components are separated from the carrier component by twice the rotation frequency of the dipole. The amplitudes of the sideband components are in general not equal and hence both amplitude and phase modulation of the scattered signal can result.

The average cross section is not affected by the rotation of the dipoles. For all polarization combinations of the transmitter and the receiver, the average cross section varies between the limits of $\frac{1}{15}\sigma_o$ and $\frac{1}{5}\sigma_o$, where σ_o is the maximum cross section. In general, the polarization of the transmitter, the receiver, or both can be varied to achieve either maximum or minimum received power for a given bistatic angle.

The correlation function of the received signal envelope from a collection of randomly rotating dipole scatterers depends upon both the transmitting and receiving polarizations, the transmitted signal envelope, and the statistical properties of the scatterers.

Chapter 3

A MODEL FOR THE RADAR ECHO FROM A RANDOM COLLECTION OF ROTATING DIPOLE SCATTERERS

3.1 Introduction

In radar systems it is often necessary to detect a target echo in the presence of other unwanted echoes or clutter. Clutter echoes include signals reflected from chaff, surface of the ground and sea, vegetation, etc. In order that a radar receiver can be designed to operate effectively in the presence of clutter interference, it is important to develop a suitable theoretical model of the clutter. Siegert [26], Kerr [13], and Lawson and Uhlenbeck [27], presented the first two probability distribution functions for the magnitude of the returned echo from an infinitely dense chaff cloud. Kelly and Lerner [24] developed a theoretical model of a chaff cloud from which they derived various statistical properties of the returned echo. The signal returned from the chaff cloud was considered as a random process and the scatterers were treated as points with variable cross sections.

The basis for the analysis can be described as follows. An

RF pulse is transmitted from a radar toward a cloud of random scatterers moving about and reflecting energy independently of one another. In addition, the cloud is assumed to have an overall drift velocity. Since the wavefront intercepts scatterers at different ranges, the echoes returned to the radar will arrive at a rate which depends upon the local density of the cloud. If the effects of multiple scattering and scatterer rotation are neglected, the echo signal from a particular scatterer can be regarded as a Doppler-shifted replica of the transmitted waveform. However, any change in the orientation of the scatterers can cause variations in the returned echo power and phase, which in general cannot be neglected. It is the purpose of the present analysis to show that rotational motion of the scatterers can have significant effects upon the echo waveform.

"Chaff" is a form of countermeasure used against radar. A chaff cloud usually consists of a large number of metallic strips (dipoles), which closely resemble an assembly of random scatterers. When chaff dipoles are dispensed from a moving craft in space, the effects of ejection forces, body instability, and other aerodynamical properties can cause the dipoles to rotate. Thus, ignoring the effects of scatterer rotation can result in an incorrect representation of the chaff model, although in many instances such effects can be tolerated without greatly affecting the results. Ground clutter exhibits echoing

characteristics similar to those of random scatterers. Rotational motion of the scatterers, or equivalent movements of branches, leaves, grass, etc., under the effects of wind forces, can become a major contributing factor to fluctuation of echo intensity. Kerr [13] and Lawson and Uhlenbeck [27] presented some experimental results on the measurements of chaff models and ground clutter. In the case of chaff measurements, it was reported that, in addition to the fast Doppler-beat fluctuations a slow secular variation was almost always observed. The origin of this type of fluctuation was not clear, but it was conjectured that the cause could be due to the rotation of dipoles. For the case of ground clutter, it was shown that the width of the fluctuation frequency spectrum was not proportional to radio frequency as would be expected if the fluctuations were due entirely to the velocity distribution of the scatterers. At some wavelengths, the fluctuation spectra approximately resembled those of randomly moving scatterers, while at other wavelengths the measured results differed significantly from the theoretically expected values. Kerr indicated that this anomalous behavior could be due to fluctuations of two sources instead of one, but no theoretical explanation was offered to clarify this phenomenon.

This chapter presents a refinement of the theoretical model for the radar echo from a random collection of scatterers. The

scatterers are treated as dipoles instead of points with variable cross sections. This assumption permits the determination of the effects of scatterer rotation which have previously been neglected [24] - [27]. The theory can be extended to the study of other classes of clutter signal which exhibit similar echoing characteristics. For example, targets making up ground clutter such as trees, grass, etc., will move and/or oscillate with the wind. The individual blades or stems scatter energy like lossy dipoles. For our present work, we ignore the loss characteristics of the scatterers. The relative magnitude of energy reflected by individual scatterers is taken care of in the probability distribution, of course.

An expression for the correlation function is derived in terms of the characteristics of the transmitted waveform, polarization, and the distribution of the scatterers. Under the assumption that the cloud density is slowly varying, the process becomes stationary. It is shown that the power spectrum of the returned echo consists of three closely spaced components, one centered at the mean Doppler frequency, while the other two are centered at the sum and difference of the mean Doppler frequency and the twice the mean rotation frequency of the dipoles. The fluctuating characteristics of clutter echoes are also determined, and the theory is compared with some of the experimental results in the literature.

3.2 Analysis of the Model

Let the coordinate system be centered at the radar. The cloud consists of a random collection of moving dipoles reflecting energy independent of one another. In addition, each dipole is assumed to have a periodic rotation about an arbitrary axis perpendicular to its length.[2] The polarization of the transmitted wave is assumed to be uniform throughout the illuminated volume and the effects of multiple scattering are neglected. As in Equations (2.37) and (2.38), assume that the transmitted signal has the form,

$$e_o(t) = \mathrm{Re}\,[\tilde{s}(t)e^{i\omega_o t}] \quad . \quad (3.1)$$

where $\tilde{s}(t)$ is the complex modulation envelope. The returned echo from a collection of scatterers located at (r_k, θ_k, ϕ_k), $k=1,2,3,\ldots$, can be written as

$$x(t) = \mathrm{Re}\,[z(t)]$$

$$z(t) = K_1 \sum_k \frac{G(\theta_k, \phi_k)}{r_k^2} \tilde{a}_k\left(t - \frac{t_k}{2}\right) \tilde{s}(t - t_k)$$

$$\cdot \exp\{i[(\omega_o + \omega_{dk})(t - t_k) + \beta_k + \epsilon(\theta_k, \phi_k)]\} \qquad (3.2$$

---

[2] The term "rotation" used here implies the instantaneous change in orientation of the dipole with respect to the direction of the radar during the time interval in which the dipole is illuminated by the radar beam. For the purpose of calculation, it is convenient to represent such a change in orientation by a periodic rotation about an appropriate axis.

where

K_1 = a constant

G = power gain of antenna[3]

$\tilde{a}_k$ = reflection coefficient

$\omega_{dk} = \left(\frac{2v_o}{c} + \frac{2v_k}{c}\right)\omega_o$ = Doppler frequency shift

v_o = overall drift radial velocity of the cloud

v_k = differential radial velocity of k^{th} scatterer

c = velocity of light

β_k = phase shift depending on initial parameters

ϵ = phase shift due to the antenna beam phase characteristics.

The geometry for the k^{th} scatterer is shown in Figure 3. The rotation axis of the dipole is described by the angles ξ_k and η_k, where ξ_k is the angle between the rotation axis and the unit vector $\hat{e}_r$ (the direction of propagation) and η_k is the angle between the projection of the rotation axis in the $\theta\phi$-plane (the plane perpendicular to the direction of propagation) and the unit vector $\hat{e}_\theta$. The reflection coefficient takes the same form as Eq. (2.40), i.e.,

[3] If the antenna has different gains for transmitting and receiving, i.e., due to polarization changes, then G should be replaced by $\sqrt{G_T G_R}$.

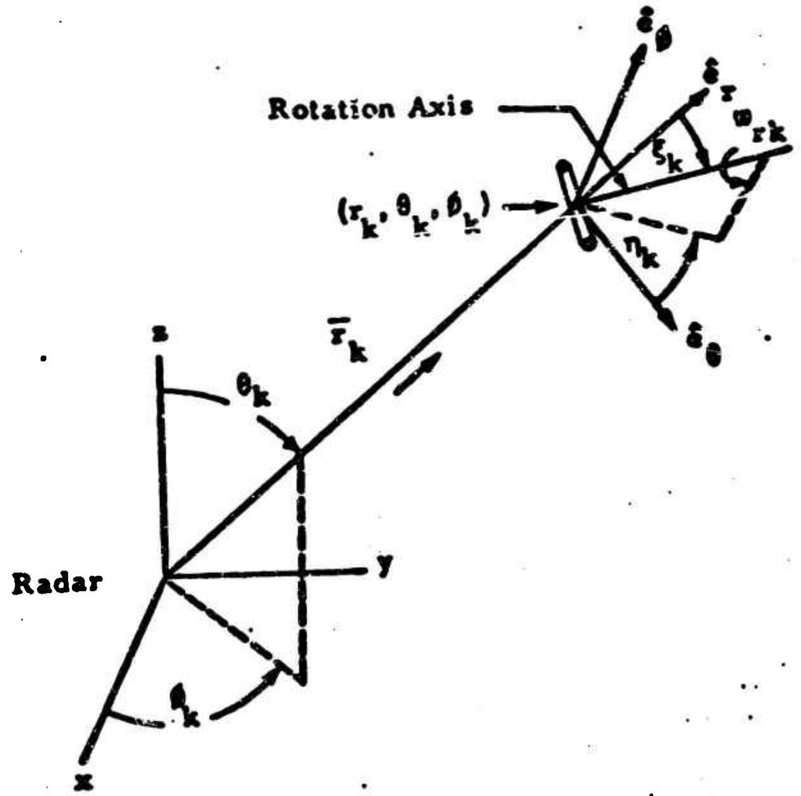

Fig. 3 - Geometry for k^{th} Dipole Scatterer

$$\tilde{a}_k(t) = \tilde{a}_{ok}\,[\tilde{C}(\xi_k, \eta_k) + \tilde{U}(\xi_k, \eta_k)\exp[+i\,2(\omega_{rk}t + \alpha_k)]$$

$$+ \tilde{L}(\xi_k, \eta_k)\exp[-i\,2(\omega_{rk}t + \alpha_k)]] \qquad (3.3)$$

where $\tilde{a}_{ok}$ is the broadside echo voltage of the dipole with the electric field vector parallel to its axis, ω_{rk} is the angular rotation frequency of the dipole, α_k is the initial angle between the dipole axis and the plane perpendicular to the direction of propagation, and the quantities $\tilde{C}$, $\tilde{U}$, and $\tilde{L}$ are functions of polarization and the orientation of the rotation axis of the dipole.

If we assume that the cloud occupies a finite volume in space and the mean distance between the cloud and the radar is large so that the difference between the range of a particular scatterer r_k and the mean range r_o is small compared with r_o, we can ignore the positional dependence of the amplitude of the echo signal. Hence, substituting (3.3) into (3.2) and setting $r_k = r_o$, we obtain

$$z(t) = \frac{K_1}{r_o^2}\sum_{k=1}^{n} G(\theta_k, \phi_k)\,\tilde{a}_{ok}\,\tilde{s}(t - t_k)\exp[+i\varphi_c(t - t_k)]$$

$$\cdot \left\{\tilde{C}(\xi_k, \eta_k) + \tilde{U}(\xi_k, \eta_k)\exp[+i\varphi_s(t - \frac{t_k}{2})]\right.$$

$$\left.+ \tilde{L}(\xi_k, \eta_k)\exp[-i\varphi_s(t - \frac{t_k}{2})]\right\} \qquad (3.4)$$

where

$$\varphi_c(t) = (\omega_o + \omega_{dk})t + \beta_k + \epsilon(\theta_k, \phi_k) \tag{3.5}$$

$$\varphi_s(t) = 2(\omega_{rk}t + \alpha_k) \tag{3.6}$$

The same statistical assumptions as in Section 2.4, Chapter 2, can be made regarding the scatterers. The random values $\tilde{a}_{ok}$, θ_k, ϕ_k, ξ_k, η_k, β_k, α_k, ω_{dk}, and ω_{rk} are taken from the joint probability distribution where it is presumed that a scatterer is present at range r_k and time $t_k/2$. It is shown in Appendix II that the probability distribution of the echo waveform approaches Gaussian as the rate of echo arrivals becomes large.

Following the same procedure as in Chapter 2, the received signal correlation function is found to be

$$R(\tau) = K_2 \nu_o e^{i\omega_o\tau} \psi_{\omega_d}(\tau) M(\tau) \int_{-\infty}^{+\infty} \tilde{s}^*(u-\tau)\,\tilde{s}(u)\,du \tag{3.7}$$

where

$$K_2 = \frac{1}{2r_o^4} |K_1|^2 \langle G^2 \rangle \langle |\tilde{a}_o|^2 \rangle \tag{3.8}$$

ν_o = average echo rate

$$M(\tau) = [\langle C^2 \rangle + \langle U^2 \rangle \psi_{\omega_r}(2\tau) + \langle L^2 \rangle \psi_{\omega_r}(-2\tau)] \tag{3.9}$$

$$\phi_{\omega_d}(\tau) = \int_{-\infty}^{+\infty} p(\omega_d) e^{i\omega_d \tau} d\omega_d$$

$$= \text{characteristic function of } p(\omega_d) \quad (3.10)$$

$$\phi_{\omega_r}(\tau) = \int_{-\infty}^{+\infty} p(\omega_r) e^{i\omega_r \tau} d\omega_r$$

$$= \text{characteristic function of } p(\omega_r) \quad (3.11)$$

and $<C^2>$, $<U^2>$, and $<L^2>$ are the expected values of $|\tilde{C}|^2$, $|\tilde{U}|^2$, and $|\tilde{L}|^2$, respectively. If we define $\rho(\tau) = R(\tau)/R(o)$ where $R(o)$ is the noise power, (3.7) becomes

$$\rho(\tau) = \frac{e^{i\omega_o \tau}}{<b_o^2>} A(\tau) \phi_{\omega_d}(\tau) M(\tau) \quad (3.12)$$

where $<b_o^2> = <C^2 + U^2 + L^2>$ and

$$A(\tau) = \frac{\int_{-\infty}^{+\infty} \tilde{s}^*(t-\tau)\tilde{s}(t)\, dt}{\int_{-\infty}^{+\infty} |\tilde{s}(t)|^2 dt} \quad (3.13)$$

is the normalized autocorrelation of the transmitted signal envelope. The power spectrum of the echo waveform can be obtained by taking the Fourier transform of (3.12). Thus,

$$g(\omega) = \frac{1}{<b_o^2>} [<C^2> F_c(\omega-\omega_o) + <U^2> F_u(\omega-\omega_o) + <L^2> F_\ell(\omega-\omega_o)] \tag{3.14}$$

where

$$F_c(\omega) = \int_{-\infty}^{+\infty} p(\omega_d)\, f(\omega-\omega_d)\, d\omega_d \tag{3.15}$$

$$F_u(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(\omega_d)\, p(\omega_r)\, f(\omega-\omega_d-2\omega_r)\, d\omega_d\, d\omega_r \tag{3.16}$$

$$F_\ell(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(\omega_d)\, p(\omega_r)\, f(\omega-\omega_d+2\omega_r)\, d\omega_d\, d\omega_r \tag{3.17}$$

and

$$f(\omega) = \int_{-\infty}^{+\infty} A(\tau)\, e^{-i\omega\tau}\, d\tau \tag{3.18}$$

is the power spectrum of the transmitted signal envelope. The power spectrum is symmetric if $p(\omega_d)$ and $p(\omega_r)$ are symmetric. For example, if $p(\omega_d)$ and $p(\omega_r)$ are Gaussian with means $\bar{\omega}_d$ and $\bar{\omega}_r$, the power spectrum can be interpreted as consisting of three closely spaced components, one centered at the mean Doppler frequency $(\omega_o + \bar{\omega}_d)$ while the other two are centered at $(\omega_o + \bar{\omega}_d + 2\bar{\omega}_r)$ and $(\omega_o + \bar{\omega}_d - 2\bar{\omega}_r)$.

It is noted that the normalized correlation function and power

spectrum do not depend on polarization if the effects of scatterer rotation are neglected. Thus, if $\omega_r = 0$, (3.12) and (3.14) reduce to

$$\rho_o(\tau) = A(\tau)\phi_{\omega_d}(\tau)e^{i\omega_o\tau} \qquad (3.19)$$

$$g_o(\omega) = F_c(\omega-\omega_o) \qquad (3.20)$$

These results are equivalent to those obtained by others [24], [25], provided that the echo rate is nearly uniform.

The average backscattered power contained in the three spectral components can be evaluated by using Equations (2.51) through (2.60). Since the scattered power is polarization dependent, the results for the following special cases are summarized below:

(1) linear transmit - linear receive

(2) circular transmit - circular receive

(3) orthogonal[4] linear transmit and receive

(4) orthogonal circular transmit and receive

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[4] This implies that the transmitter and receiver polarization are orthogonal to each other, i.e., $\hat{e}_T \cdot \hat{e}_R^* = 0$.

Case	$<C^2>$	$<U^2>$	$<L^2>$	$<b_o^2>$
(1)	$\frac{2}{15}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{5}$
(2)	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{2}{15}$
(3)	$\frac{1}{60}$	$\frac{1}{40}$	$\frac{1}{40}$	$\frac{1}{15}$
(4)	$\frac{7}{60}$	$\frac{1}{120}$	$\frac{1}{120}$	$\frac{2}{15}$

(3.21)

Other cases can be evaluated in a similar fashion.

3.3 Characteristics of Echo Fluctuations

The echo power fluctuates with time because of the motion of the scatterers. For simplicity, we assume in the following analysis that the mean number of scatterers at a given range is large so that the process becomes essentially Gaussian. The first and second probability distribution for the echo power are thus given by [13], [26] - [27].

$$W_1(P)\,dP = e^{-\frac{P}{P_o}}\,\frac{dP}{P_o} \tag{3.22}$$

$$W_2(P_1, P_2, \tau)\,dP_1 dP_2 = e^{-\frac{P_1+P_2}{P_0(1-\rho'^2)}}\, I_0\left(\frac{2\rho'\sqrt{P_1P_2}}{P_0(1-\rho'^2)}\right)\frac{dP_1 dP_2}{P_0^2(1-\rho'^2)} \tag{3.23}$$

where I_0 is the zeroth order modified Bessel function of the first kind, P_0 is the average echo power, $P_1 = P(t_1)$, $P_2 = P(t_2)$, $\tau = t_2 - t_1$, and ρ' is the correlation coefficient between the echoes of successive pulses. From (3.12), we obtain

$$\rho'(\tau) = \frac{1}{\langle b_o^2 \rangle} \operatorname{Re}\left[e^{i\omega_o \tau} \phi_{\omega_d}(\tau) M(\tau)\right] \tag{3.24}$$

The correlation function for P can be defined as

$$I(\tau) = \frac{\langle (P_1 - P_0)(P_2 - P_0) \rangle}{\langle P_1^2 \rangle - P_0^2} = \frac{\langle P_1 P_2 \rangle - P_0^2}{\langle P_1^2 \rangle - P_0^2} \tag{3.25}$$

Using (3.20) and (3.21), we obtain

$$\langle P_1^2 \rangle = \int_0^\infty W_1(P_1) P_1^2 \, dP_1 = 2P_0^2$$

$$\langle P_1 P_2 \rangle = \int_0^\infty \int_0^\infty W_2(P_1, P_2, \tau) P_1 P_2 \, dP_1 \, dP_2 = P_0^2 (1 + \rho'^2)$$

Thus, (3.25) becomes

$$I(\tau) = [\rho'(\tau)]^2 \tag{3.26}$$

Now, we assume that both the Doppler shift and the rotation frequency have a Gaussian probability distribution with means $\bar{\omega}_d$ and $\bar{\omega}_r$,

respectively. Thus,

$$p(\omega_d) = \frac{1}{\sqrt{2\pi}\,\sigma_d} \exp\left[-(\omega_d-\bar{\omega}_d)^2/2\sigma_d^2\right] \tag{3.27}$$

$$p(\omega_r) = \frac{1}{\sqrt{2\pi}\,\sigma_r} \exp\left[-(\omega_r-\bar{\omega}_r)^2/2\sigma_r^2\right] \tag{3.28}$$

From (3.10) and (3.11), the characteristic functions of $p(\omega_d)$ and $p(\omega_r)$ become,

$$\phi_{\omega_d}(\tau) = e^{i\bar{\omega}_d\tau}\, e^{-\frac{1}{2}\sigma_d^2\tau^2} \tag{3.29}$$

$$\phi_{\omega_r}(\tau) = e^{i\bar{\omega}_r\tau}\, e^{-\frac{1}{2}\sigma_r^2\tau^2} \tag{3.30}$$

Substituting (3.9), (3.29) and (3.30) in (3.24) and suppressing the phase factor $e^{i(\omega_o+\bar{\omega}_d)\tau}$ (which corresponds to mixing the returned signal with an oscillator of frequency $\omega_o+\bar{\omega}_d$), then (3.26) becomes

$$I(\tau) = e^{-\sigma_d^2\tau^2}\left[\frac{<C^2>}{<b_o^2>} + e^{-2\sigma_r^2\tau^2}\left(\frac{<U^2>}{<b_o^2>} + \frac{<L^2>}{<b_o^2>}\right)\cos 2\bar{\omega}_r\tau\right]^2 \tag{3.31}$$

Since $<U^2> = <L^2> = <S^2>$, (3.31) can be written as

$$I(\tau) = e^{-\sigma_d^2\tau^2}\left[\frac{<C^2>}{<b_o^2>} + 2\,\frac{<S^2>}{<b_o^2>}\,e^{-2\sigma_r^2\tau^2}\cos 2\bar{\omega}_r\tau\right]^2 \tag{3.32}$$

The frequency spectrum of the power fluctuations can be obtained by taking the Fourier transform of $I(\tau)$. Thus,

$$G(\Omega) = \frac{\sqrt{\pi}}{\sigma_d}\left(\frac{<C^2>}{<b_o^2>}\right)^2\left\{e^{-\frac{\Omega^2}{4}} + 2\,\frac{\sigma_d}{\sigma_a}\left(\frac{<S^2>}{<C^2>}\right)^2 e^{-\frac{1}{4}\left(\frac{\sigma_d}{\sigma_a}\right)^2\Omega^2}\right.$$
$$+ 2\,\frac{\sigma_d}{\sigma_b}\left(\frac{<S^2>}{<C^2>}\right)^2\left[e^{-\frac{1}{4}\left(\frac{\sigma_d}{\sigma_b}\right)^2\left(\Omega - 2\frac{\bar{\omega}_r}{\sigma_d}\right)^2} + e^{-\frac{1}{4}\left(\frac{\sigma_d}{\sigma_b}\right)^2\left(\Omega + 2\frac{\bar{\omega}_r}{\sigma_d}\right)^2}\right]$$
$$\left. + \frac{\sigma_d}{\sigma_a}\left(\frac{<S^2>}{<C^2>}\right)^2\left[e^{-\frac{1}{4}\left(\frac{\sigma_d}{\sigma_a}\right)^2\left(\Omega - 4\frac{\bar{\omega}_r}{\sigma_r}\right)^2} + e^{-\frac{1}{4}\left(\frac{\sigma_d}{\sigma_a}\right)^2\left(\Omega + 4\frac{\bar{\omega}_r}{\sigma_d}\right)^2}\right]\right\} \tag{3.33}$$

where

$$\Omega = \omega/\sigma_d \tag{3.34}$$

$$\sigma_a^2 = \sigma_d^2 + 4\sigma_r^2 \tag{3.35}$$

$$\sigma_b^2 = \sigma_d^2 + 2\sigma_r^2 \tag{3.36}$$

One of our objectives was to clarify theoretically some of the anomalous phenomena observed during early experimental studies of

clutter [13]. Figure 4 illustrates the correlation function for wooded ground clutter measured at 9.2 cm and at wind speeds of 50 mph. The experimental data were taken from Kerr [13, p. 587]. The theoretical values are shown by the dashed curve, which can be obtained from (3.32) and (3.21) with $\langle C^2\rangle / \langle b_o^2\rangle = \frac{2}{3}$ and $\langle S^2\rangle / \langle b_o^2\rangle = \frac{1}{6}$ for linear polarization. Thus, we have

$$I(\tau) = e^{-0.0289 \times 10^4 \tau^2} \cdot \left[\frac{2}{3} + \frac{1}{3} e^{-0.309 \times 10^4 \tau^2} \cos (25\pi)\tau\right]^2 . \qquad (3.37)$$

This corresponds to $\sigma_d = 17$ rad/s and $\bar{\omega}_r = \sigma_r = 39.3$ rad/s. The corresponding frequency spectrum of the fluctuations can be obtained by taking the Fourier transform of the correlation function, and is shown graphically in Figure 5.

It should be mentioned that the measured correlation functions were obtained by radar envelope measurements. However, Kerr [13] has indicated that the difference between the correlation functions for envelope and power is small, the maximum deviation being 0.027. Hence, by neglecting this small difference, the two correlations can be considered equivalent.

It is noted that the curves deviate significantly from the usual expected Gaussian shape. Instead, they drop very slowly toward

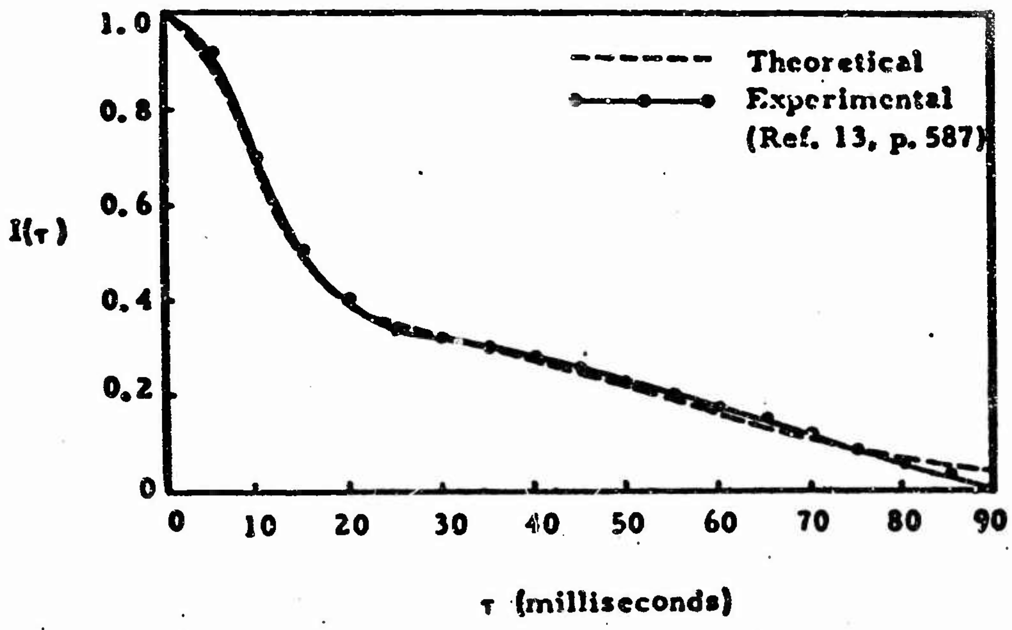

Fig. 4 - Correlation Function for Ground Clutter on 9.2 cm (Heavily Wooded Terrain at Wind Speeds of 50 mph)

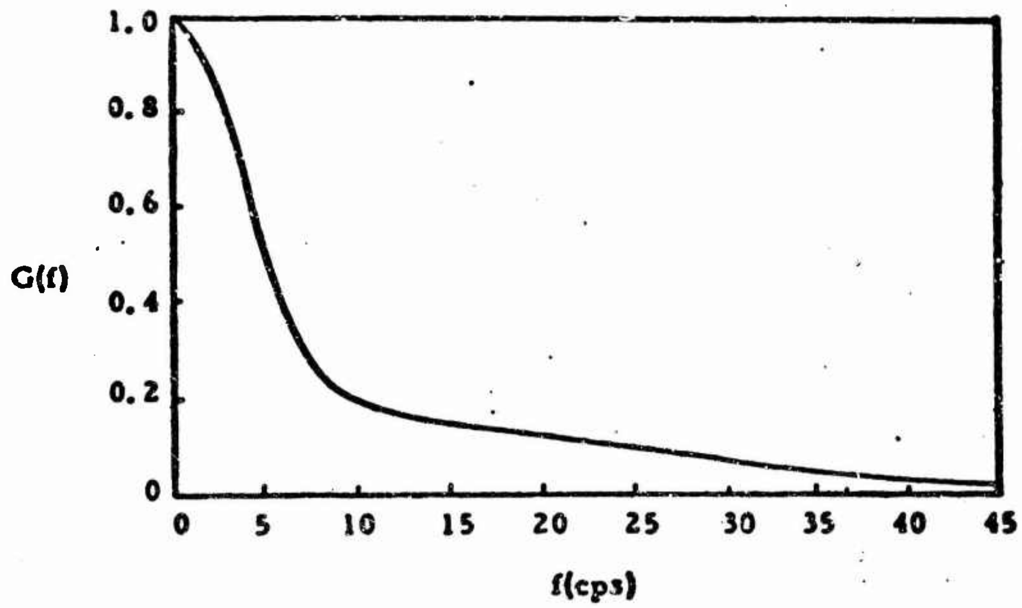

Fig. 5 - Fluctuation Frequency Spectrum for Ground Clutter on 9.2 cm (Heavily Wooded Terrain at Wind Speeds of 50 mph)

zero. The anomalous appearance of the correlation function is due predominantly to the oscillating character of the scatterers. Kerr has pointed out that a small positive asymptote has been subtracted from the original measured correlation function in an attempt to correct for the slow secular variations. The correlation function has been renormalized to a value of 1.0 at $\tau = 0$. Hence, an exact comparison between the calculated and measured values cannot be made. However, the close resemblance between the shapes of the theoretical and experimental curves indicates good agreement of theory with experiment.

3.4 Conclusions

When a radar operates in an environment of clutter, it is important to determine all possible effects which can limit the effectiveness of the receiver. If the echo waveform were precisely known, it would be possible, in principle, to synthesize a filter which optimizes the signal-to-noise ratio at the output of the receiver. In this chapter, we have emphasized the effects of scatterer rotation. The following conclusions can be drawn.

1) If the spectral spread due to the Doppler velocity fluctuations of the scatterers is large, then the effects due to scatterer rotation can be neglected. Receiver

characteristics will be primarily determined by the Doppler spectrum of the clutter.

2) If the Doppler spread is small, then the contributions due to rotations of the scatterers can become significant. This was demonstrated in the example considered at the end of Section 3.3.

Chapter 4

ON SCATTER COMMUNICATION VIA RANDOM TIME-VARIANT "DISCRETE SCATTERER" CHANNEL

4.1 Introduction

In many radar applications, communication may be established between two distant points by means of electromagnetic scattering from a collection of scatterers randomly distributed in a region of space. An elementary scatter communication system is shown in Fig. 6. The region of space occupied by the collection of scatterers is referred to as the scattering medium. An example of such a medium is a cloud of resonant dipoles (chaff) [7]-[11]. The basic equation relating the system parameters is the bistatic radar equation,

$$\frac{P_R}{P_T} = \frac{G_T\, G_R\, \lambda^2}{(4\pi)^3\, r_T^2\, r_R^2}\, \sigma \tag{4.1}$$

where the subscripts R and T refer to the receiver and transmitter parameters, respectively, and the quantity σ is the scattering function which depends upon the characteristics of the medium.

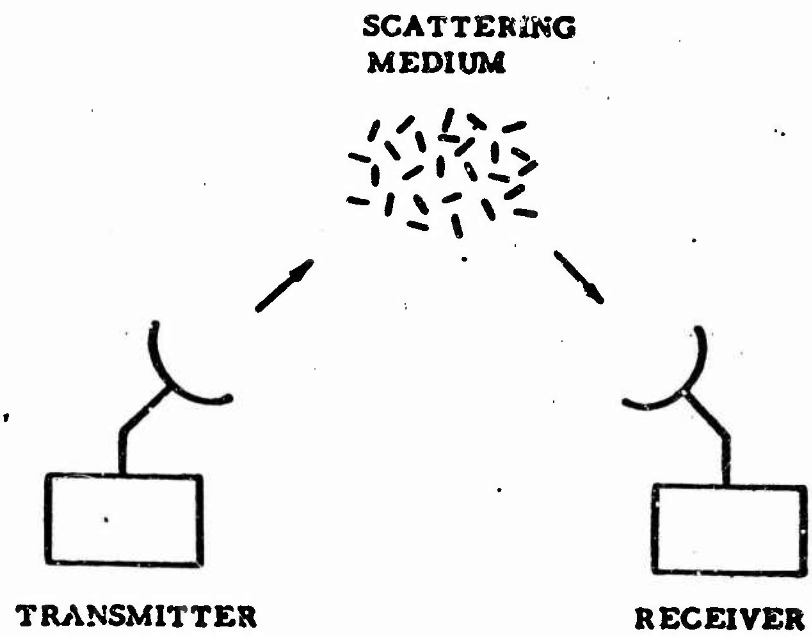

Figure 6. Scatter Communication System

In general the scattering medium varies with both position and time. When viewed from the transmitting and receiving terminals, the scatter channel acts like a linear, random, time-varying four-terminal network. Our objective is to develop a suitable response function for this network.

In the analyses of random scattering, most investigators ignore the time dependence and are concerned only with the mean power which may be provided by the channel with a sine-wave excitation. For example, in Eq. (4.1), σ usually represents the average cross section of the scattering medium. In a practical satuation, the medium is time-varying, the transmitted waveform is a modulated carrier, and the received signal is passed through various filters. The design of such filters requires more information than the mean channel power output or the average cross section of the scattering medium.

The present analysis is an extension to that of Kelly [14]. The general theory of electromagnetic scattering from a random collection of small metallic scatterers is developed. The antenna gain functions are included in the derivation of the channel function in order to take explicit account of the possibility that scatterers may flow in and out of the volume illuminated by the two antenna beams. Arbitrary transmitting and receiving polarizations are considered,

and the effects of scatterer rotation are also taken into account. Finally, specific results are obtained for the case of dipole scattering.

4.2 Derivation of the Scattered Field

Consider the scattering from a collection of metallic scatterers which are contained within a finite volume V. Assume that the origin of coordinates is also contained within V and that all sources which produce the primary field are external to this volume. The fields everywhere satisfy the Maxwell's equations,

$$\begin{aligned} \nabla \times \bar{E} &= -\mu_o \frac{\partial \bar{H}}{\partial t} \\ \nabla \times \bar{H} &= \epsilon_o \frac{\partial \bar{E}}{\partial t} + \bar{J} \end{aligned} \tag{4.2}$$

where $\bar{J}$ is the source current which gives rise to the primary field. If the scatterers are perfectly conducting, the boundary conditions $\hat{n} \cdot \bar{H} = 0$ and $\hat{n} \times \bar{E} = 0$ are satisfied on the surface of each scatterer. It follows from (4.2) that $\bar{E}$ satisfies the vector Helmholtz equation,

$$\nabla \times \nabla \times \bar{E} + \mu_o \epsilon_o \frac{\partial^2 \bar{E}}{\partial t^2} = -\mu_o \frac{\partial \bar{J}}{\partial t} \tag{4.3}$$

We assume that the time dependence of the fields and its sources can be resolved into Fourier components,

$$\bar{E}(\bar{r},t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{E}(\bar{r},\omega) e^{+i\omega t} \, d\omega, \text{ etc.} \tag{4.4}$$

Equation (4.3) becomes

$$\nabla \times \nabla \times \bar{E} - k^2\bar{E} = -i\omega\mu_o\bar{J} \tag{4.5}$$

where $\bar{E} = \bar{E}(\bar{r},\omega)$, $\bar{J} = \bar{J}(\bar{r},\omega)$, and $k^2 = \omega^2\mu_o\epsilon_o$.

Application of the vector Green's theorem, together with the proper boundary conditions and the radiation condition for $r \to \infty$, to (4.5) yields (Appendix III)

$$\bar{E}(\bar{r}) = \frac{k^2}{i\omega\epsilon_c}\left\{\int_{V_o} \bar{\bar{\Gamma}}(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}')\,dV' + \int_S \bar{\bar{\Gamma}}(\bar{r},\bar{r}'') \cdot \hat{n} \times \bar{H}(\bar{r}'')\,dS''\right\} \tag{4.6}$$

where

$$\bar{\bar{\Gamma}}(\bar{r},\bar{r}_o) = \left(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla\right)\frac{e^{-ik|\bar{r}-\bar{r}_o|}}{4\pi|\bar{r}-\bar{r}_o|} \tag{4.7}$$

is the dyadic free space Green's function [28] and $\bar{\bar{I}}$ is the identity dyadic. Other field quantities are related by Maxwell's equations. The first term on the right side of (4.6) is the primary field, $\bar{E}_o(\bar{r})$. Hence, the scattered field is given by

$$\bar{E}_s(\bar{r}) = \frac{k^2}{i\omega\epsilon_o}\int_S \bar{\bar{\Gamma}}(\bar{r},\bar{r}'') \cdot \hat{n} \times \bar{H}(\bar{r}'')\,dS'' \tag{4.8}$$

where the integration is carried over the surface of the scatterer.

The geometry for a single scatterer is shown in Fig. 7. We

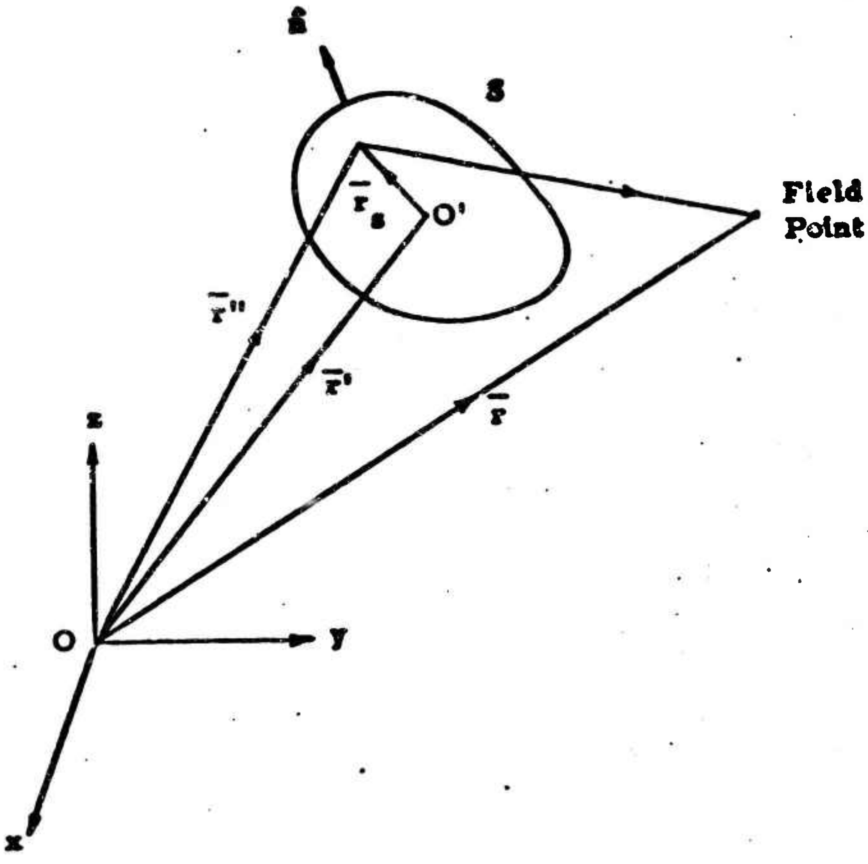

Figure 7. Coordinate System for a Single Scatterer

let $\overline{r}'$ be the vector distance from the coordinate origin to the center of the scatterer, and $\overline{r}''$ be the vector distance from the coordinate origin to a point on the surface of the scatterer. In addition, we let $\overline{r}_s = \overline{r}''-\overline{r}'$. Then, in the far zone, $|\overline{r}-\overline{r}''| \approx |\overline{r}-\overline{r}'| - \hat{a}' \cdot \overline{r}_s$, where $\hat{a}'$ is a unit vector in the $(\overline{r}-\overline{r}')$ direction, and

$$\overline{\overline{\Gamma}}(\overline{r},\overline{r}'') \approx \frac{e^{-ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}(\overline{\overline{I}} - \hat{a}'\hat{a}')\, e^{+ik\hat{a}'\cdot\overline{r}_s} \tag{4.9}$$

Equation (4.8) becomes

$$\overline{E}_s(\overline{r}) = \frac{-k^2}{i\omega\epsilon_o}\, \frac{e^{-ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}\, \hat{a}'\times\hat{a}'\times \int_S [\hat{n}\times \overline{H}(\overline{r}'')]\, e^{+ik\hat{a}'\cdot\overline{r}_s}\, dS'' \tag{4.10}$$

If the effects of multiple scattering due to the presence of other scatterers are neglected and if the scatterer dimension is small compared to the wavelength, the surface current induced on the scatterer may be approximated by that due to the primary field alone. If $k|\overline{r}_s|$ is sufficiently small, the exponential inside the integral may be replaced by $(1 + ik\hat{a}' \cdot \overline{r}_s)$. With these approximations, equation (4.10) becomes

$$\overline{E}_s(\overline{r}) = -\frac{k^2}{i\omega\epsilon_o}\, \frac{e^{-ik|\overline{r}-\overline{r}'|}}{4\pi|\overline{r}-\overline{r}'|}$$

$$\hat{a}'\times\hat{a}'\times\int_S [\hat{n}\times\overline{H}_o(\overline{r}'')](1 + ik\hat{a}'\cdot\overline{r}_s)\, dS'' \tag{4.11}$$

Next it can be shown that, [29]

$$\int_S (\hat{a}'\cdot\bar{r}_s)[\hat{n}\times\bar{H}_o(\bar{r}'')]\,dS'' = -\frac{1}{2}\hat{a}'\times\int_S \bar{r}_s\times[\hat{n}\times\bar{H}_o(\bar{r}'')]\,dS'' \quad . \tag{4.12}$$

Hence

$$\bar{E}_s(\bar{r}) = -\frac{k^2}{4\pi\epsilon_o}\frac{e^{-ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\left[\hat{a}'\times(\hat{a}'\times\bar{p}) + \frac{k}{\omega}\hat{a}'\times\bar{m}\right] \tag{4.13}$$

where $\bar{p}$ and $\bar{m}$ are the induced electric and magnetic dipole moments, respectively:

$$\bar{p} = \frac{1}{i\omega}\int_S [\hat{n}\times\bar{H}_o(\bar{r}'')]\,dS'' \tag{4.14}$$

$$\bar{m} = \frac{1}{2}\int_S \bar{r}_s\times[\hat{n}\times\bar{H}_o(\bar{r}'')]\,dS'' \tag{4.15}$$

Thus, in the far zone, the scatterer acts as though it were both an electric and a magnetic dipole in a uniform static field. In classical electromagnetism the induced electric and magnetic dipole moments are also expressed as linear vector functions of the applied fields $\bar{E}_o(\bar{r}')$ and $\bar{H}_o(\bar{r}')$, respectively [30]- [31], as follows:

$$\bar{p} = \epsilon_o\bar{\bar{\alpha}}\cdot\bar{E}_o(\bar{r}') \tag{4.16}$$

$$\bar{m} = \bar{\bar{\beta}}\cdot\bar{H}_o(\bar{r}') \tag{4.17}$$

where $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ are the dyadic electric and magnetic polarizabilities which depend on the size, shape, and orientation of the scatterer.

Now assume that the scatterers are located at points $\overline{r}'_i$ where $i = 1, 2, \ldots, n$. By Eq. (4.13) with $\overline{p}$ and $\overline{m}$ given by Equations (4.16) and (4.17), respectively, the total scattered field becomes

$$\overline{E}_s(\overline{r}) = -\frac{k^2}{4\pi} \sum_{i=1}^{n} \frac{e^{-ik|\overline{r}-\overline{r}'_i|}}{|\overline{r}-\overline{r}'_i|}$$

$$\cdot \left\{ \hat{a}'_i \times [\hat{a}'_i \times \overline{\overline{\alpha}}_i \cdot \overline{E}_o(\overline{r}'_i)] + \frac{k}{\omega\epsilon_o} \hat{a}'_i \times \overline{\overline{\beta}}_i \cdot \overline{H}_o(\overline{r}'_i) \right\} \qquad (4.18)$$

In the far zone, $\overline{r}$ is far from V so we may put $\hat{a}'_i = \hat{a}_r$ and $|\overline{r}-\overline{r}'_i| = r - \hat{a}_r \cdot \overline{r}'_i$. With this substitution, (4.18) becomes

$$\overline{E}_s(\overline{r},\omega) = -\frac{\omega^2}{4\pi c^2} \frac{e^{-i\frac{\omega}{c}r}}{r} \sum_{i=1}^{n} e^{+i\frac{\omega}{c}\hat{a}_r \cdot \overline{r}'_i}$$

$$\left\{ \hat{a}_r \times [\hat{a}_r \times \overline{\overline{\alpha}}_i \cdot \overline{E}_o(\overline{r}'_i,\omega)] + \frac{1}{\epsilon_o c} \hat{a}_r \times \overline{\overline{\beta}}_i \cdot \overline{H}_o(\overline{r}'_i,\omega) \right\} \qquad (4.19)$$

where we have replaced k^2 by ω^2/c^2.

In order to put (4.19) in the form of a stochastic integral, we use the method of E. J. Kelly [14]. Kelly assumes that the position, size, shape, and orientation of any scatterer can be represented by a set of parameters, $x_1, x_2, \ldots, x_\nu$, which we take to be the coordinates of a point $\vec{x}$ in a ν-dimensional space X. This space is a

direct product of the ordinary three-space V, which describes the position of the scatterers, and a parameter space Y, whose coordinates fix the size, shape, and orientation of the scatterers. Thus, specification of a point $\vec{x}$ in the one scatterer phase space is represented by $\vec{x} = (\vec{r}, \vec{y})$, where $\vec{r}$ locates the scatterer and $\vec{y}$ describes all other parameters. If $\vec{x}_i$ locates the i^{th} scatterer in its phase space, one writes $\overline{\overline{\alpha}}_i = \overline{\overline{\alpha}}(\vec{x}_i)$, $\overline{\overline{\beta}}_i = \overline{\overline{\beta}}(\vec{x}_i)$, and $\overline{r}'_i = \overline{r}'(\vec{x}_i)$, where $\overline{\overline{\alpha}}(\vec{x})$ and $\beta(\vec{x})$ depend on all but the spatial components of $\vec{x}$ and $\overline{r}'(\vec{x})$ has components equal to the positional coordinates included in $\vec{x}$. Next we introduce an integral-valued measure, N(S), of measurable sets in the ν-dimensional phase space X, which describes the collection of scatterers. If S is a measurable set in X, we take N(S) to be the number of scatterers for which $\vec{x}_i$ belongs to S. In this manner, Eq. (4.19) may be written in the form of the Stieltje's integral,

$$\overline{E}(\overline{r}, \omega) = -\frac{\omega^2}{4\pi c^2} \frac{e^{-i\frac{\omega}{c} r}}{r} \int_X e^{+i\frac{\omega}{c} \hat{a}_r \cdot \overline{r}'_i(\vec{x})}$$

$$\cdot \left\{ \hat{a}_r \times [\hat{a}_r \times \overline{\overline{\alpha}}(\vec{x}) \cdot \overline{E}_o(\overline{r}'(\vec{x}))] + \frac{1}{\epsilon_o c} \hat{a}_r \times \overline{\overline{\beta}}(\vec{x}) \cdot \overline{H}_o(\overline{r}'(\vec{x}), \omega) \right\} dN(\vec{x}) \tag{4.20}$$

Using Eq. (4.4) or taking the Fourier transform, we find

$$\bar{E}_s(\bar{r},t) = -\frac{1}{8\pi^2 c^2 r}\int_X dN(\vec{x})\int_{-\infty}^{+\infty}\omega^2\left\{\hat{a}_r \times [\hat{a}_r \times \bar{\bar{\alpha}}(\vec{x})\cdot \bar{E}_o(\bar{r}'(\vec{x}),\omega)]\right.$$

$$\left. + \frac{1}{\epsilon_o c}\,\hat{a}_r \times \bar{\bar{\beta}}(\vec{x})\cdot \bar{H}_o(\bar{r}'(\vec{x}),\omega)\right\} e^{i\omega[t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot \bar{r}'(\vec{x})]}\, d\omega \qquad (4.21)$$

Since

$$-\omega^2 \bar{E}_o(\bar{r},\omega) = \int_{-\infty}^{+\infty} \ddot{\bar{E}}_o(\bar{r},t)\, e^{-i\omega t} dt$$

Equation (4.21) becomes

$$\bar{E}_s(\bar{r},t) = \frac{1}{4\pi c^2 r}\int_X \left\{\hat{a}_r \times [\hat{a}_r \times \bar{\bar{\alpha}}(\vec{x})\cdot \ddot{\bar{E}}_o(\bar{r}'(\vec{x}), t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot \bar{r}'(\vec{x}))]\right.$$

$$\left. + \frac{1}{\epsilon_o c}\,\hat{a}_r \times \bar{\bar{\beta}}(\vec{x})\cdot \ddot{\bar{H}}_o(\bar{r}'(\vec{x}), t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot \bar{r}'(\vec{x}))\right\} dN(\vec{x}) \qquad (4.22)$$

If a probability measure is impressed on the sets of X, Eq. (4.22) can be treated as a stochastic integral.

If the scatterers are slowly moving or changing their size, shape, and orientation with time, the measure N then becomes time-dependent. If we assume that all these motions are slow with respect to the period of the primary impressed field, Eq. (4.22) will remain valid with $dN(\vec{x})$ replaced by $dN(\vec{x},t)$.

4.3 Derivation of the Channel Function

Assume that a transmitter is located at a vector distance $\bar{r}_T$ from the origin and that the collection of scatterers, which produces the scattering, is in the far zone of the transmitting antenna. If a real signal f(t) is applied to the input terminals of the transmitter, then, in the scattering region, the transmitted fields can be represented by

$$\bar{E}_o(\bar{r},t) = K_o\sqrt{G_T(\bar{r})}\ \hat{e}_T f\left[t + \frac{1}{c}\hat{a}_T\cdot(\bar{r}-\bar{r}_T)\right]$$
$$\bar{H}_o(\bar{r},t) = \frac{K_o}{\eta_o}\sqrt{G_T(\bar{r})}\ \hat{h}_T f\left[t + \frac{1}{c}\hat{a}_T\cdot(\bar{r}-\bar{r}_T)\right] \tag{4.24}$$

where

$$K_o = \left(\frac{\eta_o}{4\pi r_T^2}\right)^{1/2} \tag{4.25}$$

$\eta_o = \sqrt{\mu_o/\epsilon_o}$ is the intrinsic impedance of free space, G_T is the gain function of the transmitting antenna, $\hat{a}_T$ is a unit vector in the $\bar{r}_T$ direction, $\hat{e}_T$ is the electric polarization vector, and $\hat{h}_T = -\hat{a}_T \times \hat{e}_T$ (the wave is propagating in the $-\hat{a}_T$ direction). Substituting (4.24) in (4.22), we obtain

$$\bar{E}_s(\bar{r},t) = \frac{K_o}{4\pi c^2 r}\int_X [G_T(\bar{r}'(\vec{x}))]^{1/2}\left\{\hat{a}_r \times [\hat{a}_r \times \bar{\bar{\alpha}}(\vec{x})\cdot \hat{e}_T] + \hat{a}_r \times \bar{\bar{\beta}}(\vec{x})\cdot \hat{h}_T\right\}$$

$$\ddot{f}\left[t - \frac{1}{c}(r+r_T) + \frac{1}{c}(\hat{a}_T + \hat{a}_r)\cdot \bar{r}'(\vec{x})\right] dN(\vec{x},t) \qquad (4.26)$$

Next, we assume that a receiver is located at a vector distance $\bar{r}_R$, also far from the scattering region. Then, for $\bar{r}$ near $\bar{r}_R$, we have $1/r \approx 1/r_R$, $\hat{a}_r \approx \hat{a}_R$, and $r \approx r_R + \hat{a}_R\cdot(\bar{r}-\bar{r}_R)$, where $\hat{a}_R$ is a unit vector in the $\bar{r}_R$ direction. Equation (4.26) becomes

$$\bar{E}_s(\bar{r},t) = \int_X [G_T(\bar{r}'(\vec{x}))]^{1/2}\, \bar{w}(\vec{x})$$

$$\ddot{f}\left[t - t_o - \frac{1}{c}(\hat{a}_T + \hat{a}_R)\cdot \bar{r}'(\vec{x}) - \frac{1}{c}(\bar{r}-\bar{r}_R)\cdot \hat{a}_R\right] dN(\vec{x},t) \qquad (4.27)$$

where

$$t_o = \frac{1}{c}(r_T + r_R)$$

is the total time delay from transmitter to origin to receiver, and

$$\bar{w}(\vec{x}) = \frac{K_o}{4\pi c^2 r_R}\left\{\hat{a}_R \times [\hat{a}_R \times \bar{\bar{\alpha}}(\vec{x})\cdot \hat{e}_T] + \hat{a}_R \times \bar{\bar{\beta}}(\vec{x})\cdot \hat{h}_T\right\} \qquad (4.28)$$

Now, Equation (4.27) can be put in the form

$$\overline{E}_s(\overline{r},t) = \int_X \int_{-\infty}^{+\infty} [G_T(\overline{r}'(\vec{x}))]^{1/2}\, \overline{w}(\vec{x})\ddot{f}\,[t-s-\frac{1}{c}(\overline{r}-\overline{r}_R)\cdot\hat{a}_R]$$

$$\delta[s-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\overline{r}'(\vec{x})]\,ds\,dN(\vec{x},t) \qquad (4.29)$$

where $\delta(\cdot)$ is the Dirac delta function. In this form $\overline{E}_s$ represents a superposition of plane waves, propagating in the direction of the receiving antenna, with slowly varying amplitudes. Let $\hat{e}_R$ be the polarization vector of the receiving antenna and G_R be its gain function. Then, by flux considerations, the plane wave

$$\sqrt{G_T}\;\overline{w}\,g[t-\frac{1}{c}(\overline{r}-\overline{r}_R)\cdot\hat{a}_R]$$

propagating over the receiving antenna, will produce a signal

$$\frac{\lambda}{\sqrt{4\pi\eta_o}}\sqrt{G_TG_R}\;(\hat{e}_R\cdot\overline{w})\,g(t)$$

at the antenna terminals, where G_R is the gain of the receiving antenna in the $-\hat{a}_R$ direction. Since the scatterers are in motion, the scattered signal will in general have a time-varying amplitude and phase. If we assume that the receiving antenna is sufficiently wideband to follow these variations, the total received signal will be

$$h(t) = \frac{\lambda}{\sqrt{4\pi \eta_o}} \int_X \int_{-\infty}^{+\infty} [G_T(\overline{r}'(\vec{x}))G_R(\overline{r}'(\vec{x}))]^{1/2} [\hat{e}_R \cdot \overline{w}(\vec{x})] \ddot{f}(t-s)$$

$$\delta [s-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot \overline{r}'(\vec{x})] ds \, dN(\vec{x},t) \qquad (4.30)$$

For convenience, we let

$$[G_T(\overline{r}')G_R(\overline{r}')]^{1/2} = [G_{OT}G_{OR}]^{1/2} G(\overline{r}') \qquad (4.31)$$

where G_{OT} and G_{OR} are the maximum gain of the transmitting and receiving antennas, respectively, and $G(\overline{r}')$ is the normalized gain function relating the product of the two antenna gains. Substituting (4.28) and (4.31) in (4.30) and using the value of K_o from (4.25), we find

$$h(t) = \frac{-A_o}{\sqrt{4\pi c^2}} \int_X \int_{-\infty}^{+\infty} G(\overline{r}'(\vec{x}))[\hat{e}_R \cdot \overline{\overline{\alpha}}(\vec{x}) \cdot \hat{e}_T + \hat{h}_R \cdot \beta(\vec{x}) \cdot \hat{h}_T]$$

$$\ddot{f}(t-s)\, \delta[t-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot \overline{r}'(\vec{x})] ds \, dN(\vec{x},t) \qquad (4.32)$$

where

$$A_o^2 = \frac{G_{OT}G_{OR}\lambda^2}{(4\pi)^3 r_T^2 r_R^2} \qquad (4.33)$$

and we have made use of the fact that $\hat{e}_R \cdot \hat{a}_R = 0$ and $\hat{h}_R = \hat{a}_R \times \hat{e}_R$.

If the overall scattering process is regarded as a random time-varying channel, we may put

$$h(t) = \int_0^{\infty} k(t,s)\, f(t-s)\, ds \tag{4.34}$$

where $k(t,s)$ is the response function of the channel. If

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega \tag{4.35}$$

then

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K(t,\omega) F(\omega)\, e^{i\omega t}\, d\omega \tag{4.36}$$

where

$$K(t,\omega) = \int_0^{\infty} k(t,s)\, e^{-i\omega s}\, ds \tag{4.36}$$

Following Kelly, we will call $K(t,\omega)$ the channel function. From (4.32), (4.35), and (4.36), the channel function becomes

$$K(t,\omega) = A_o e^{-i\omega t_o} \int_X G(\vec{r}'(\vec{x}))Q(\omega,\vec{x}) e^{i\frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot\vec{r}'(\vec{x})}\, dN(\vec{x},t) \tag{4.37}$$

where

$$Q(\omega,\vec{x}) = \frac{\omega^2}{\sqrt{4\pi c^2}} \left[\hat{e}_R \cdot \bar{\bar{\alpha}}(\vec{x}) \cdot \hat{e}_T + \hat{h}_R \cdot \bar{\bar{\beta}}(\vec{x}) \cdot \hat{h}_T \right] \tag{4.38}$$

This is a generalization of a similar result obtained by Kelly [14]. In (4.37) the collection of scatterers behaves as it were a dispersive medium. The response of the medium is described by the channel function. In the stationary case N is independent of time, and the frequency response is proportional to ω^2 since our analysis has been

based on the static polarizabilities of the scatterers. If the scatterers are not small, our procedure remains valid if we allow the polarizabilities to depend on frequency. In this case, Eq. (4.37) still holds with $\overline{\overline{\alpha}}(\vec{x})$ and $\overline{\overline{\beta}}(\vec{x})$ replaced by $\overline{\overline{\alpha}}(\omega,\vec{x})$ and $\overline{\overline{\beta}}(\omega,\vec{x})$.

4.4 Example: Dipole Scattering

Consider the scattering from a collection of axially symmetric scatterers (i.e., cylindrical dipoles). The electric and magnetic polarizability dyadics for axially symmetric scatterers can be represented by

$$\begin{aligned} \overline{\overline{\alpha}} &= \alpha_1 \hat{d}\hat{d} + \alpha_2 \hat{\rho}\hat{\rho} \\ \overline{\overline{\beta}} &= \beta_1 \hat{d}\hat{d} + \beta_2 \hat{\rho}\hat{\rho} \end{aligned} \tag{4.39}$$

where $\hat{d}$ and $\hat{\rho}$ are unit vectors parallel and perpendicular to the axis of the scatterer, respectively. The components of the dyadics $\overline{\overline{\alpha}}$ and $\overline{\overline{\beta}}$ are complex function of frequency and the size and shape of the scatterer. The transmitting and receiving polarization vectors can be written as

$$\begin{aligned} \hat{e}_T &= (\hat{d}\cdot\hat{e}_T)\hat{d} + (\hat{\rho}\cdot\hat{e}_T)\hat{\rho} \\ \hat{e}_R &= (\hat{d}\cdot\hat{e}_R)\hat{d} + (\hat{\rho}\cdot\hat{e}_R)\hat{\rho} \end{aligned} \tag{4.40}$$

Hence

$$\hat{e}_R \cdot \overline{\overline{\alpha}} \cdot \hat{e}_T = [(\alpha_1 - \alpha_2)(\hat{d} \cdot \hat{e}_R)(\hat{d} \cdot \hat{e}_T) + \alpha_2(\hat{e}_R \cdot \hat{e}_T)] \tag{4.41}$$

and

$$\hat{h}_R \cdot \overline{\overline{\beta}} \cdot \hat{h}_T = [(\beta_1 - \beta_2)(\hat{d} \cdot \hat{h}_R)(\hat{d} \cdot \hat{h}_T) + \beta_2(\hat{h}_R \cdot \hat{h}_T)] \tag{4.42}$$

For thin dipoles, the magnitudes of α_2, β_1, and β_2 are small compared with α_1. Then

$$Q(\omega, \vec{x}) \approx \frac{\omega^2 \alpha_1}{\sqrt{4\pi}\, c^2} (\hat{d} \cdot \hat{e}_R)(\hat{d} \cdot \hat{e}_T) \tag{4.43}$$

and the channel function becomes

$$K(t, \omega) = \frac{A_o \omega^2}{\sqrt{4\pi}\, c^2} e^{-i\omega t_o} \int_X e^{+i\frac{\omega}{c}(\hat{a}_T + \hat{a}_R) \cdot \vec{r}'(\vec{x})}$$

$$[\alpha_1(\omega, \vec{x})(\hat{d} \cdot \hat{e}_R)(\hat{d} \cdot \hat{e}_T)]\, dN(\vec{x}, t) \tag{4.44}$$

where $\alpha_1(\omega, \vec{x})$ depends on all but the orientation coordinates included in $\vec{x}$.

Rotational motion of the scatterers can be taken into account by allowing the orientation vector $\hat{d}$ to become time-dependent. It has been shown in Chapter 2 that the quantity $(\hat{d} \cdot \hat{e}_R)(\hat{d} \cdot \hat{e}_T)$ may be put in the form

$$(\hat{d} \cdot \hat{e}_R)(\hat{d} \cdot \hat{e}_T) = [\tilde{C} + \tilde{U} e^{+i2\phi} + \tilde{L} e^{-i2\phi}] \tag{4.45}$$

where $\tilde{C}$, $\tilde{U}$, and $\tilde{L}$ are functions of the orientation of the rotation axis of the dipole, and ψ is the instantaneous orientation angle of the dipole axis. Comparison between (2.28) and (4.43) shows that the quantity $|Q(\omega,\vec{x})|^2$ has the significance of the cross section of an individual dipole. If all orientations of the rotation axis are equally likely, the average cross section is given by

$$< |Q(\omega,\vec{x})|^2 > = \sigma_o < b_o^2 > \tag{4.46}$$

where σ_o is the maximum cross section, and

$$< b_o^2 > = < C^2 + U^2 + L^2 >$$

which has been evaluated in Chapter 2 (Eq. 2.35). In the present chapter, the bistatic angle β (the angle between the directions of the transmitter and receiver, viewed from the scattering region) is given by

$$\cos\beta = \hat{a}_T \cdot \hat{a}_R \tag{4.47}$$

4.5 Statistical Properties

In the previous sections, we have derived results for the scattering from a definite set of scatterers in space. In this section, we will examine some of the statistical properties of the channel output.

The collection of scatterers is described by an integral-valued measure, $N(S,t)$, of sets in the one-scatterer phase space X. Since the scattering is random in nature, $N(S,t)$ becomes a randomly time-varying random measure of sets in X. Poisson statistics are assumed for $N(S,t)$ so that the random variables $N(S_1,t),\ldots,N(S_n,t)$ are independent if the sets $S_1,\ldots,S_n$ are disjoint, and that the probability that $N(S,t)$ equals the integer n is given by

$$P(n) = \frac{[<N(S,t)>]^n}{n!} \exp[-<N(S,t)>] \tag{4.48}$$

where $<N(S,t)> = E\,N(S,t)$ denotes the average number[5] of scatterers in set S at time t. We assume that the measure $<N(S,t)>$ has a density $<n(\vec{x},t)>$ so that

$$<N(S,t)> = \int_S <n(\vec{x},t)>\,dX \tag{4.49}$$

where dX is an infinitesimal element of volume in the phase space X. If we let

$$w_i = \int_X W_i(\vec{x})\,dN(\vec{x},t)\,, \quad i=1,2. \tag{4.50}$$

then

$$E\,w_i = \int_X W_i(\vec{x}) <n(\vec{x},t)>\,dX \tag{4.51}$$

[5] The symbol E also stands for the expected value or ensemble average. We will use these two notations interchangeably.

$$\text{Var } w_i = \int_X W_i^2(\vec{x}) < n(\vec{x}, t) > dX \tag{4.52}$$

and

$$\text{Cov } [w_1, w_2] = \int_X W_1(\vec{x}) W_2(\vec{x}) < n(\vec{x}, t) > dX \tag{4.53}$$

Now, from (4.34) and (4.36), the channel response function may be expressed in two useful forms,

$$\begin{aligned} h(t) &= \int_0^\infty k(t, s)\, f(t-s)\, ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} K(t, \omega) F(\omega)\, e^{i\omega t}\, d\omega \end{aligned} \tag{4.54}$$

where

$$k(t, s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K(t, \omega)\, e^{i\omega s}\, d\omega \tag{4.55}$$

we find

$$\begin{aligned} E\, h(t) &= \int_0^\infty E\, k(t, s)\, f(t-s)\, ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\, K(t, \omega) F(\omega)\, e^{i\omega t}\, d\omega \end{aligned} \tag{4.56}$$

and

$$\begin{aligned} \text{Cov}[h(t), h(t')] &= \int_0^\infty \int_0^\infty \text{Cov}[k(t, s), k(t', s')]\, f(t-s) f(t'-s')\, ds\, ds' \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \text{Cov}[K(t, \omega), K^*(t', \omega')]\, e^{i(\omega t - \omega' t')} F(\omega) F^*(\omega')\, d\omega\, d\omega' \end{aligned} \tag{4.57}$$

The two channel covariances are related by

$$\mathrm{Cov}\,[k(t,s),k(t',s')] = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{Cov}\,[K(t,\omega),K^*(t',\omega')]\,e^{i(\omega s-\omega' s')}\,d\omega\,d\omega' \tag{4.58}$$

Equation (4.56) implies that the mean output to f(t) is determined by the mean properties of the channel. From (4.37) and (4.51), we obtain

$$E\,K(t,\omega) = A_o e^{-i\omega t_o}\int_X e^{i\frac{\omega}{c}(\hat{a}_T+\hat{a}_R)\cdot\overline{r}'(\vec{x})}\, G(\overline{r}'(\vec{x}))\,Q(\omega,\vec{x})\,<n(\vec{x},t)>\,dX \tag{4.59}$$

Recall that $\vec{x} = (\vec{r},\vec{y})$ so that $dX = dV\,dY$, and $Q(\omega,\vec{x})$ depends only on the y-parameters. Furthermore, in many situations, it is possible to put

$$<n(\vec{x},t)> = f(\vec{y})<n(\overline{r},t)> \tag{4.60}$$

where $<n(\overline{r},t)>$ is the mean number density in the ordinary space, and $f(\vec{y})$ is the probability density on the y-parameters, which may also be time-dependent. Then, Equation (4.59) can be written in the form

$$E\,K(t,\omega) = A_o e^{-i\omega t_o}<Q(\omega,\vec{y})> \cdot\int_V e^{i\frac{\omega}{c}(\hat{a}_T+\hat{a}_R)\cdot\overline{r}'}\,G(\overline{r}')<n(\overline{r}',t)>dV' \tag{4.61}$$

where

$$< Q(\omega,\vec{y}) > = \int_Y f(\vec{y})\, Q(\omega,\vec{y})\, dY \tag{4.62}$$

In the dipole scattering example of Section 4.4, if we assume a uniform probability of all orientations at any one time, we find

$$<(\hat{d}\cdot\hat{e}_R)(\hat{d}\cdot\hat{e}_T)> = \frac{1}{3}(\hat{e}_R\cdot\hat{e}_T)$$

and

$$< Q(\omega,\vec{y}) > = \frac{1}{3}\,\frac{\omega^2}{\sqrt{4\pi}\,c^2}\,\alpha_1(\omega)(\hat{e}_R\cdot\hat{e}_T) \tag{4.63}$$

where $\hat{e}_R$ and $\hat{e}_T$ must refer to the same coordinate system. In terms of the notations used in Section 2.2 of Chapter 2, we put

$$\hat{e}_T = \tilde{a}_r\hat{\mathcal{R}} + \tilde{a}_\ell\hat{\mathcal{L}}$$

$$\hat{e}_R = \tilde{a}'_r\hat{\mathcal{R}}' + \tilde{a}'_\ell\hat{\mathcal{L}}'$$

Here $\hat{e}_T$ and $\hat{e}_R$ are the polarization vectors referred to the transmitter and receiver coordinate systems, respectively. With the aid of the rotational transformation given in Eq. (2.6), we compute

$$\left(\hat{e}_R\cdot\hat{e}_T\right) = \left[(\tilde{a}_\ell\tilde{a}'_r + \tilde{a}_r\tilde{a}'_\ell)\cos^2\frac{\beta}{2} - (\tilde{a}_\ell\tilde{a}'_\ell + \tilde{a}_r\tilde{a}'_r)\sin^2\frac{\beta}{2}\right] \tag{4.64}$$

This expression can easily be evaluated once the transmitter and receiver polarizations and the bistatic angle are known.

The covariance of the channel output can now be investigated.

Since the scatterers are in motion, we begin by examining the flow of the variables $\vec{x} = (\vec{r},\vec{y})$ in the phase space X. Consider two elements $dN(\vec{x},t)$ and $dN(\vec{x}',t')$. Suppose that $\vec{x}$ flows into $\vec{x}_1$ as time goes from t to $\bar{t}$ and $\vec{x}_2$ flows into $\vec{x}'$ as time goes from $\bar{t}$ to t', where $\bar{t} = (t+t')/2$. For sufficiently small values of t-t', we may put

$$\vec{x} = \vec{x}_1 - \dot{\vec{x}}_1\left(\frac{t'-t}{2}\right) \tag{4.65}$$

and

$$\vec{x}' = \vec{x}_2 + \dot{\vec{x}}_2\left(\frac{t'-t}{2}\right) \tag{4.66}$$

where $\dot{\vec{x}}$ is the rate of change of the position vector $\vec{x}$ in the one scatterer phase space. Then

$$w = \int_X W(\vec{x})\,dN(\vec{x},t) = \int_X W\left[\vec{x}_1 - \dot{\vec{x}}_1\left(\frac{t'-t}{2}\right)\right]dN(\vec{x}_1,\bar{t})$$

$$w' = \int_X W(\vec{x}')dN(\vec{x}',t') = \int_X W\left[\vec{x}_2 + \dot{\vec{x}}_2\left(\frac{t'-t}{2}\right)\right]dN(\vec{x}_2,\bar{t})$$

where $\dot{\vec{x}}_1 = \dot{\vec{x}}_1(\vec{x}_1,\bar{t})$ and $\dot{\vec{x}}_2 = \dot{\vec{x}}_2(\vec{x}_2,\bar{t})$. It follows from the assumption of Poisson statistics that

$$\mathrm{Cov}[w,w'] = \int_X E\, W\left[\vec{x}-\dot{\vec{x}}\left(\frac{t'-t}{2}\right)\right]W\left[\vec{x}+\dot{\vec{x}}\left(\frac{t'-t}{2}\right)\right] < n(\vec{x},t) > dX \tag{4.67}$$

where we have assumed that the velocity field $\dot{\vec{x}}(\vec{x},t)$ is statistically

independent of the measure $N(S, t)$.

Using (4.57) and making assumption (4.60), we obtain

$$
\begin{aligned}
\mathrm{Cov}\left[K(t,\omega), K^*(t',\omega')\right] &= A_o^2 \exp\left[-i(\omega-\omega')t_o\right] \\
&\int_V \int_Y E\, G\left[\bar{r}' - \dot{\bar{r}}'\left(\frac{t'-t}{2}\right)\right] \exp\left\{i \frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot\left[\bar{r}' - \dot{\bar{r}}'\left(\frac{t'-t}{2}\right)\right]\right\} \\
&G^*\left[\bar{r}' + \dot{\bar{r}}'\left(\frac{t'-t}{2}\right)\right] \exp\left\{-i \frac{\omega'}{c}(\hat{a}_T + \hat{a}_R)\cdot\left[\bar{r}' + \dot{\bar{r}}'\left(\frac{t'-t}{2}\right)\right]\right\} \\
&Q\left[\omega, \vec{y} - \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] Q^*\left[\omega', \vec{y} + \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] f(\vec{y}) < n(\bar{r}', \bar{t}) > d^3\bar{r}'\, dY \\
&= A_o^2 \exp\left[-i(\omega-\omega')t_o\right] \int_V \exp\left[\frac{i}{c}(\omega-\omega')(\hat{a}_T + \hat{a}_R)\cdot \bar{r}'\right] \\
&E\left\{\exp\left[-\frac{i}{c}(\omega+\omega')\left(\frac{t'-t}{2}\right)(\hat{a}_T + \hat{a}_R)\cdot \bar{v}(\bar{r}', \bar{t})\right]\right. \\
&\left. G\left[\bar{r}' - \bar{v}(\bar{r}', \bar{t})\left(\frac{t'-t}{2}\right)\right] G^*\left[\bar{r}' + \bar{v}(\bar{r}', \bar{t})\left(\frac{t'-t}{2}\right)\right]\right\} < n(\bar{r}', \bar{t}) > d^3\bar{r}' \\
&\int_Y E\, Q\left[\omega, \vec{y} - \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] Q^*\left[\omega', \vec{y} + \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] f(\vec{y})\, dY
\end{aligned}
\tag{4.68}
$$

where $\bar{v}(\bar{r}, t) = \dot{\bar{r}}(\bar{r}, t)$ and we have assumed that $\bar{v}$ and $\dot{\vec{y}}$ are independent. Let $F(\bar{v}; \bar{r}, t)$ be the probability density that a scatterer has a velocity $\bar{v}$ at $(\bar{r}, t)$. Eq. (4.68) may be written as

$$\mathrm{Cov}\left[K(t,\omega),K^*(t',\omega')\right] = A_o^2 \exp\left[-i(\omega-\omega')t_o\right]$$

$$\int_Y E\, Q\left[\omega, \vec{y} - \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] Q^*\left[\omega', \vec{y} + \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] f(\vec{y})\, dY$$

$$\int\int\left\{ \exp\left[\frac{i}{c}(\omega-\omega')(\hat{a}_T + \hat{a}_R)\cdot \bar{r}'\right]\right\} F(\bar{v}; \bar{r}', \bar{t})$$

$$\left\{G\left[\bar{r}' - \bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right] G^*\left[\bar{r}' + \bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]\right.$$

$$\left.\exp\left[-\frac{i}{c}(\omega+\omega')\left(\frac{t'-t}{2}\right)(\hat{a}_T + \hat{a}_R)\cdot \bar{v}(\bar{r}',\bar{t})\right]\right\} < n(\bar{r}',\bar{t}) > d^3\bar{v}\, d^3\bar{r}' \tag{4.69}$$

We assume that $Q(\omega,\vec{y})$ is sufficiently wideband, so that $Q(\omega,\vec{y}) \approx Q(\omega_o,\vec{y})$ over the signal bandwidth, where ω_o is the angular carrier frequency. In addition we assume that $\frac{v}{c}(t'-t)B << 1$, where B is the signal bandwidth. Substituting (4.69) in (4.57), we obtain

$$\mathrm{Cov}\left[h(t),h(t')\right] = \frac{A_o^2}{(2\pi)^2} W(\omega_o, t'-t)$$

$$\int\int\left\{\int_{-\infty}^{+\infty} F(\omega)\exp\left[i\omega\left\{t - t_o + \frac{1}{c}(\hat{a}_T + \hat{a}_R)\cdot\left[\bar{r}' - \bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]\right\}\right] d\omega\right.$$

$$\left.\int_{-\infty}^{+\infty} F^*(\omega')\exp\left[-i\omega'\left\{t' - t_o + \frac{1}{c}(\hat{a}_T + \hat{a}_R)\left[\bar{r}' + \bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]\right\}\right] d\omega'\right\}$$

$$G\left[\bar{r}' - \bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right] G^*\left[\bar{r}' + \bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]$$

$$F(\bar{v}; \bar{r}', \bar{t}) < n(\bar{r}',\bar{t}) > d^3\bar{v}\, d^3\bar{r}' \tag{4.70}$$

where

$$W(\omega_o, t'-t) = \int_Y E\, Q\left[\omega_o, \vec{y} - \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] Q^*\left[\omega_o, \vec{y} + \dot{\vec{y}}\left(\frac{t'-t}{2}\right)\right] f(\vec{y})\, dY \tag{4.71}$$

Again the method and results given above are a slight generation of those given by E. J. Kelly in Reference [14].

For the dipole scattering example, the $\vec{y}$-parameters include ξ, η, and ψ (see Fig. 2), where the angles ξ and η define the orientation of the rotation axis and $\psi = \omega_r t + \alpha$ is the instantaneous orientation angle of the dipole with respect to the u-axis. Here $\omega_r = \dot{\psi}$ and α is the initial orientation angle. If we assume that every orientation of the rotation axis is equally probable and that the initial orientation of the dipole axis is uniformly distributed in the interval $0 \leq \alpha \leq 2\pi$, we find

$$W(\omega_o, \tau) = \frac{\omega_o^4}{\sqrt{4\pi}\, c^4} |\alpha_1(\omega_o)|^2 \left[< C^2 > + < U^2 > \phi_{\omega_r}(2\tau) + < L^2 > \phi_{\omega_r}(-2\tau) \right] \tag{4.72}$$

where $\tau = t'-t$ and $\phi_{\omega_r}(\tau)$ is the characteristic function of $p(\omega_r)$ defined in Chapter 2.

Since

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

and $f(t)$ is real so that $F^*(-\omega) = F(\omega)$. Equation (4.70) may be written as

$$\mathrm{Cov}\,[h(t),h(t')] = A_o^2\, W(\omega_o, t'-t)\cdot$$

$$\int\int f\left\{t-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[\bar{r}'-\bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]\right\}\cdot$$

$$f\left\{t'-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[\bar{r}'+\bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]\right\}$$

$$G\left[\bar{r}'-\bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]G^*\left[\bar{r}'+\bar{v}(\bar{r}',\bar{t})\left(\frac{t'-t}{2}\right)\right]$$

$$F(\bar{v};\bar{r}',\bar{t}) < n(\bar{r}',\bar{t}) > d^3\bar{v}\; d^3\bar{r}' \tag{4.73}$$

Now f(t) can be written in the form of a modulated carrier,

$$f(t) = m(t)\cos\left[\omega_o t + \varphi_o(t)\right] \tag{4.74}$$

Substitute (4.74) in (4.73) and assume that m(t) and $\varphi_o(t)$ vary only slowly compared to cos $\omega_o t$. Then, by keeping t'-t sufficiently small, we may drop the term $\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)$ from the arguments of m(t) and $\varphi_o(t)$. We then use the sum and difference identity on the product of cosines and drop the sum term because of its rapid variation with $\bar{r}'$. Equation (4.73) becomes

$$\mathrm{Cov}\,[h(t),h(t')] = \frac{1}{2} A_o^2\, W(\omega_o, t'-t)$$

$$\iint G\left[\overline{r}'-\overline{v}(\overline{r}',\overline{t})\left(\frac{t'-t}{2}\right)\right] G^*\left[\overline{r}'+\overline{v}(\overline{r}',\overline{t})\left(\frac{t'-t}{2}\right)\right]$$

$$m[t-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\overline{r}']\, m[t'-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\overline{r}']$$

$$\cos\left[\frac{\omega_o}{c}(\hat{a}_T+\hat{a}_R)\cdot\overline{v}(\overline{r}',t)(t'-t)\right] < n(\overline{r}',\overline{t}) > F(\overline{v};\overline{r}',\overline{t})\, d^3\overline{v}\, d^3\overline{r}' \tag{4.75}$$

The variance of the channel output can be deduced directly from (4.75). Thus

$$\mathrm{Var}\, h(t) = \mathrm{Cov}\,[h(t),\, h(t)]$$

$$= \frac{1}{2} A_o^2\, W(\omega_o) \int_V |G(\overline{r}')|^2 m^2[t-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\overline{r}'] < n(\overline{r}',\overline{t})> d^3\overline{r} \tag{4.76}$$

where

$$W(\omega_o) = < |Q(\omega_o,\vec{y})|^2 > = \int_Y |Q(\omega_o,\vec{y})|^2 f(\vec{y})\, dY \tag{4.77}$$

The quantity $\frac{1}{2} m^2(t)$ is proportional to the average power being transmitted at t. Likewise, Var h(t) is proportional to the average power in the fluctuating component of h(t), received at t. Recall that

$$A_o^2 = \frac{\lambda^2 G_T G_R}{(4\pi)^3 r_T^2 r_R^2}$$

hence, the quantity

$$W(\omega_o) \int_V |G(\bar{r}')|^2 < n(\bar{r}', t) > d^3\bar{r}'$$

has the significance of channel cross section.

For the case of backscattering, both $\hat{a}_T$ and $\hat{a}_R$ are in the direction of the antenna. However, the antenna may have different polarization for transmitting and receiving. If one introduces a coordinate system (x, y, z) with the z-axis in the negative $(\hat{a}_T + \hat{a}_R)$ direction, and assume that the antenna gain is nearly uniform over the cloud of scatterers, then the cross section per unit range is given by

$$\sigma(z', t) = W(\omega_o) \iint < n(x', y', z'; t) > dx' \, dy' \qquad (4.78)$$

This represents simple incoherent scattering from a collection of independent scatterers, i.e., the total cross section is equal to the average cross section per scatterer times the mean number of scatterers illuminated at time t.

For the dipole scattering example, we have

$$W(\omega_o) = \frac{\omega_o^4}{4\pi c^4} |\alpha_1(\omega_o)|^2 < C^2 + U^2 + L^2 > \qquad (4.79)$$

Chapter 5

SCATTERING BY RANDOMLY VARYING CONTINUOUS MEDIA WITH APPLICATION TO COMMUNICATIONS

5.1 Introduction

Interest in the problem of establishing long distance communications via forward scatter has motivated the study of scattering from a perturbed region of space whose electromagnetic properties vary randomly with space and time. In the previous chapter, we studied the properties of a "discrete scatterer" channel, in which the scattering medium is made up of a collection of metallic scatterers randomly distributed in a region of space. The scattering characteristics were expressed in terms of the electric and magnetic polarizabilities of each scatterer.

In the present chapter, the properties of a "continuum" channel are examined. The constitutive parameters of the medium are assumed to vary randomly with both position and time. The scattering characteristics are expressed in terms of the electric and magnetic susceptibilities of the medium. General time dependence is assumed. An expression for the channel function is derived in

terms of the scattering properties of the medium. The approaches being taken here are parallel to those of Kelly [14]; however, slightly different techniques are employed to derive the scattered fields. Also, the effect of antenna gain is explicitly taken into account in order to allow for the fact that scatterers may flow in and out of the volume illuminated by the two antenna beams. The theory is developed in a general manner so that it may be applied to the study of a wide class of problems; e.g., scattering by a turbulent and inhomogeneous atmosphere, ionized gases, and turbulent wakes.

5.2 Scattering by Continuous Media

Consider a scattering volume which is filled by a random continuous medium. The interaction of matter and the fields are described by

$$\bar{D} = \epsilon_o \bar{E} + \bar{P} \tag{5.1}$$

$$\bar{H} = \frac{1}{\mu_o} \bar{B} - \bar{M} \tag{5.2}$$

where $\bar{P}$ and $\bar{M}$ are the electric and magnetic polarization vectors, respectively. Outside the scattering volume, we have free space, and the polarization vectors vanish. The electromagnetic fields everywhere are governed by Maxwell's equations [37],

$$\nabla \times \bar{B} - \mu_o \epsilon_o \frac{\partial \bar{E}}{\partial t} = \mu_o \left(\bar{J} + \frac{\partial \bar{P}}{\partial t} + \nabla \times \bar{M} \right) \tag{5.3}$$

$$\nabla \times \bar{E} + \frac{\partial \bar{B}}{\partial t} = 0 \tag{5.4}$$

$$\nabla \cdot \bar{E} = \frac{1}{\epsilon_o} (\rho - \nabla \cdot \bar{P}) \tag{5.5}$$

$$\nabla \cdot \bar{B} = 0 \tag{5.6}$$

where ρ and $\bar{J}$ are the charge and current densities which give rise to the primary field. The presence of the material medium is thus accounted for by the polarization charge density $-\frac{1}{\epsilon_o} \nabla \cdot \bar{P}$ and the polarization current density $\frac{\partial \bar{P}}{\partial t} + \nabla \times \bar{M}$.

The electromagnetic fields can be expressed in terms of the Hertz potential $\bar{\Pi}(\bar{r}, t)$

$$\bar{E} = \nabla \nabla \cdot \bar{\Pi} - \mu_o \epsilon_o \frac{\partial^2 \bar{\Pi}}{\partial t^2} \tag{5.7}$$

$$\bar{B} = \mu_o \epsilon_o \nabla \times \frac{\partial \bar{\Pi}}{\partial t}. \tag{5.8}$$

where $\bar{\Pi}$ satisfies the differential equation,

$$\nabla^2 \bar{\Pi} - \mu_o \epsilon_o \frac{\partial^2 \bar{\Pi}}{\partial t^2} = -\frac{1}{\epsilon_o} \bar{P}(\bar{r}, t) - \frac{1}{\epsilon_o} \int_o^t [\bar{J}(\bar{r}, t'') + \nabla \times \bar{M}(\bar{r}, t'')] dt'' \tag{5.9}$$

The solution can be written as

$$\overline{\Pi}(\overline{r},t) = \frac{1}{\epsilon_o}\left\{\iint \overline{P}(\overline{r}',t')G(\overline{r},t;\overline{r}',t')\,dV'\,dt'\right.$$

$$\left.+\iint\int_o^{t'} dt''[\overline{J}(\overline{r}',t'') + \nabla'\times\overline{M}(\overline{r}',t'')]G(\overline{r},t;\overline{r}',t')dV'dt'\right\} \tag{5.10}$$

where

$$G(\overline{r},t;\overline{r}',t') = \frac{\delta\left[t'-t+\frac{|\overline{r}-\overline{r}'|}{c}\right]}{4\pi|\overline{r}-\overline{r}'|} \tag{5.11}$$

is the retarded Green's function for the scalar wave equation (Appendix IV). The Hertz vector for the primary field is given by

$$\overline{\Pi}_o(\overline{r},t) = \frac{1}{\epsilon_o}\iint\left[\int_o^{t'}\overline{J}(\overline{r}',t'')dt''\right]G(\overline{r},t;\overline{r}',t')\,dV'dt' \tag{5.12}$$

Assume that all sources which produce the primary field are external to the scattering volume. Then from (5.10), the Hertz vector for the scattered field is

$$\overline{\Pi}_s(\overline{r},t) = \frac{1}{\epsilon_o}\left\{\iint \overline{P}(\overline{r}',t')G(\overline{r},t;\overline{r}',t')\,dV'\,dt'\right.$$

$$\left.+\iint\left[\int_o^{t'}\nabla'\times\overline{M}(\overline{r}',t'')dt''\right]G(\overline{r},t;\overline{r}',t')\,dV'dt'\right\} \tag{5.13}$$

where the domain of integration is the scattering volume in which $\overline{P}$ and $\overline{M}$ differs from zero. From the vector identity $\nabla'\times G\overline{M} = G\,\nabla'\times\overline{M} - \overline{M}\times\nabla' G$, we have

$$\int_S (\hat{n} \times \overline{M})G\, dS' = \int_V G\nabla' \times \overline{M}\, dV' - \int_V \overline{M} \times \nabla' G\, dV' \qquad (5.14)$$

where the integral on the left is extended over the surface enclosing the scattering volume, and $\hat{n}$ is the unit outward normal to this surface. Assume that the scattering volume remains within a finite closed surface, which is made large enough so that the surface integral in (5.14) vanishes. Equation (5.13) can be rewritten as

$$\overline{\Pi}_s(\overline{r},t) = \frac{1}{\epsilon_o} \iint \Big\{ \overline{P}(\overline{r}',t')G(\overline{r},t;\overline{r}',t') + \int_o^{t'} \overline{M}(\overline{r}',t'') \times \nabla' G(\overline{r},t;\overline{r}',t')dt'' \Big\} dV'dt' \qquad (5.15)$$

From (5.7) and the relation $\nabla G(\overline{r},t;\overline{r}',t') = -\nabla' G(\overline{r},t;\overline{r}',t')$, the scattered field becomes

$$\overline{E}_s(\overline{r},t) = \frac{1}{\epsilon_o} \iint [\overline{P}(\overline{r}',t') \cdot \nabla']\nabla' G(\overline{r},t;\overline{r}',t')\, dV'\, dt'$$
$$- \mu_o \frac{\partial^2}{\partial t^2} \iint \overline{P}(\overline{r}',t')G(\overline{r},t;\overline{r}',t')\, dV'\, dt'$$
$$- \mu_o \frac{\partial^2}{\partial t^2} \iint \Big\{ \int_o^{t'} \overline{M}(\overline{r}',t'') \times \nabla' G(\overline{r},t;\overline{r}',t')dt'' \Big\} dV'\, dt' \qquad (5.16)$$

Let the origin of the coordinates be located within V and evaluate the scattered field only in the far zone, then

$$|\bar{r}-\bar{r}'| \approx r - \hat{a}_r \cdot \bar{r}' .$$

$$G \approx \frac{1}{4\pi r}\, \delta(t'-t+\frac{r}{c}-\frac{1}{c}\hat{a}_r \cdot \bar{r}')$$

$$\bar{M} \times \nabla' G \approx \frac{1}{4\pi c r}\, (\hat{a}_r \times \bar{M})\, \delta'(t'-t+\frac{r}{c}-\frac{1}{c}\hat{a}_r \cdot \bar{r}')$$

$$[\bar{P}\cdot\nabla']\nabla' G \approx -\frac{1}{4\pi c^2 r}\, [\bar{P}\cdot \hat{a}_r]\,\hat{a}_r\; \delta''(t'-t+\frac{r}{c}-\frac{1}{c}\hat{a}_r \cdot \bar{r}') .$$

where $\hat{a}_r$ is a unit vector along the r-direction and δ' indicates differentiation with respect to its argument. Substituting these quantities in (5.16) and making use of the formula,

$$\int f(x)\, \delta^{(n)}(x-a)\, dx = (-1)^n f^{(n)}(a)$$

we obtain

$$\bar{E}(\bar{r},t) = \frac{\mu_o}{4\pi c^2 r}\left\{ \hat{a}_r \times \int_V [\hat{a}_r \times \ddot{\bar{P}}(\bar{r}', t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot\bar{r}')]\, dV' \right.$$

$$\left. + \frac{1}{c}\hat{a}_r \times \int_V \ddot{\bar{M}}(\bar{r}', t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot\bar{r}')\, dV' \right\} . \qquad (5.17)$$

where $\ddot{\bar{P}} = \frac{\partial^2}{\partial t^2}\bar{P}$ and $\ddot{\bar{M}} = \frac{\partial^2}{\partial t^2}\bar{M}$. The scattered field can be interpreted as arising from a distribution of induced electric and magnetic dipole moments per unit volume of the scattering medium. The solution will depend on the manner in which the polarization depends on the external field. If we assume that the dependence of $\bar{p}$ and $\bar{m}$ on $\bar{E}$ and $\bar{H}$, respectively, is linear, we can write

$$\overline{P}(\overline{r},t) = \epsilon_o \int_o^\infty \overline{\overline{\chi}}(\overline{r},t,s) \cdot \overline{E}(\overline{r},t-s)\, ds \tag{5.18}$$

$$\overline{M}(\overline{r},t) = \int_o^\infty \overline{\overline{\psi}}(\overline{r},t,s) \cdot \overline{H}(\overline{r},t-s)\, ds \tag{5.19}$$

where $\overline{\overline{\chi}}$ and $\overline{\overline{\psi}}$ are the dyadic electric and magnetic susceptibilities of the medium. Equation (5.18) and (5.19) also express the fact that the induced polarization must not precede in time the field which produces it. If $\overline{\overline{\chi}}(\overline{r},t,s)$ and $\overline{\overline{\psi}}(\overline{r},t,s)$ vary slowly compared to the primary field variations, then

$$\ddot{\overline{P}}(\overline{r},t) \approx \epsilon_o \int_o^\infty \overline{\overline{\chi}}(\overline{r},t,s) \cdot \ddot{\overline{E}}(\overline{r},t-s)\, ds$$

$$\ddot{\overline{M}}(\overline{r},t) \approx \int_o^\infty \overline{\overline{\psi}}(\overline{r},t,s) \cdot \ddot{\overline{H}}(\overline{r},t-s)\, ds$$

Further let us assume that the presence of the scattering medium is a small perturbation of free space conditions so that the scattered energy is small compared to that of the primary field. Then we can use the Born approximation, replacing the total fields $\overline{E}$ and $\overline{H}$ by the primary fields $\overline{E}_o$ and $\overline{H}_o$ within the integrals. With these assumptions, Equation (5.17) becomes

$$\bar{E}_s(\bar{r},t) = \frac{1}{4\pi c^2 r}\left\{\hat{a}_r \times \int_V\int_0^\infty (\hat{a}_r \times [\bar{\bar{\chi}}(\bar{r}',t,s)\cdot\ddot{\bar{E}}_o(\bar{r}',t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot\bar{r}'-s)])dV'ds\right.$$

$$\left. + \sqrt{\frac{\mu_o}{\epsilon_o}}\,\hat{a}_r \times \int_V [\bar{\bar{\mathcal{Y}}}(\bar{r}',t,s)\cdot\ddot{\bar{H}}_o(\bar{r}',t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot\bar{r}'-s)]dV'ds\right. \qquad (5.20)$$

where we have replaced $(t-\frac{r}{c}+\frac{1}{c}\hat{a}_r\cdot\bar{r}')$ by t in the arguments of $\bar{\bar{\chi}}$ and $\bar{\bar{\mathcal{Y}}}$ since the medium is slowly varying. This is the same as Kelly's result except that here $\bar{\bar{\chi}}$ and $\bar{\bar{\mathcal{Y}}}$ are dyadics instead of scalars.

5.3 The Channel Function

The channel function for the continuum case can be derived in the same manner as for the discrete scatterer case. However, because of the similarities in the formulism between the two types of scattering, a direct analogy can be established so that the results developed for the discrete scatterer case can be applied immediately to the continuum case. Comparison between (4.22) and (5.20) shows that the two expressions are equivalent if we put

$$\bar{\bar{\alpha}}(\vec{x})\ \delta(s)\ dN(\vec{x},t) \longrightarrow \bar{\bar{\chi}}(\bar{r}',t,s)\ dV'$$

$$\bar{\bar{\beta}}(\vec{x})\ \delta(s)\ dN(\vec{x},t) \longrightarrow \bar{\bar{\mathcal{Y}}}(\bar{r}',t,s)\ dV'$$

Thus, from the results of Section 4.3, Chapter 4, we find that the channel function for the continuum case is given by

$$K(t,\omega) = A_o e^{-i\omega t_o} \int_V G(\bar{r}')Q(\bar{r}',t,\omega) e^{+i\frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot\bar{r}'} dV' \tag{5.21}$$

where

$$A_o^2 = \frac{G_{OT}\, G_{OR}\, \lambda^2}{(4\pi)^3\, r_T^2\, r_R^2} \tag{5.22}$$

$$Q(\bar{r},t,\omega) = \frac{\omega^2}{\sqrt{4\pi}\, c^2} \left[\hat{e}_R \cdot \bar{\bar{X}}(\bar{r},t,\omega) \cdot \hat{e}_T + \hat{h}_R \cdot \bar{\bar{Y}}(\bar{r},t,\omega) \cdot \hat{h}_T \right] \tag{5.23}$$

$$\bar{\bar{X}}(\bar{r},t,\omega) = \int_o^\infty \bar{\bar{\chi}}(\bar{r},t,s)\, e^{-i\omega s}\, ds \tag{5.24}$$

$$\bar{\bar{Y}}(\bar{r},t,\omega) = \int_o^\infty \bar{\bar{\psi}}(\bar{r},t,s)\, e^{-i\omega s}\, ds \tag{5.25}$$

Other quantities are defined in Chapter 4.

The condition of realizability requires that the polarization be zero for $s < 0$. This implies that $\bar{\bar{X}}(\bar{r},t,\omega)$ and $\bar{\bar{Y}}(\bar{r},t,\omega)$ must be analytic in the lower half of the complex ω-plane. Since $t_o - \frac{1}{c}(\hat{a}_T + \hat{a}_R)\cdot\bar{r}' \geq 0$ for any point in the scattering volume, it follows that $K(t,\omega)$ is also analytic in the lower half ω-plane.

For simplicity, we restrict our discussions to a scattering medium which contains only a single constituent, since the generalizations to the case for which the medium contains several constituents are straight forward. For a continuous medium with one constituent, having number density $n(\bar{r},t)$, we may put

$$\bar{\bar{X}}(\bar{r},t,\omega) = \bar{\bar{X}}(\omega)\, n(\bar{r},t)$$

$$\bar{\bar{Y}}(\bar{r},t,\omega) = \bar{\bar{Y}}(\omega)\, n(\bar{r},t)$$

where $\bar{\bar{X}}(\omega)$ and $\bar{\bar{Y}}(\omega)$ describe the dispersive properties of the medium. Thus, the channel function can be written in the form

$$K(t,\omega) = A_o Q(\omega)\, e^{-i\omega t_o} \int_V G(\bar{r}')\, e^{i\frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot \bar{r}'}\, n(\bar{r}',t)\, dV' \qquad (5.26)$$

where

$$Q(\omega) = \frac{\omega^2}{\sqrt{4\pi}\, c^2} \left[\hat{e}_R \cdot \bar{\bar{X}}(\omega) \cdot \hat{e}_T + \hat{h}_R \cdot \bar{\bar{Y}}(\omega) \cdot \hat{h}_T \right] \qquad (5.27)$$

As an example, we consider the scattering by an ionized gas (plasma). We assume that the medium is non-magnetic so that $\bar{\bar{Y}}(\bar{r},t,\omega) = 0$. The electric susceptibility takes the form [40]

$$\bar{\bar{X}}(\bar{r},t,\omega) = -\frac{e^2}{m\epsilon_o} \frac{n(\bar{r},t)}{\omega(\omega - i\nu)} \bar{\bar{I}} \qquad (5.28)$$

where $\bar{\bar{I}}$ is the identity dyadic, e and m are the electron charge and mass, $n(\bar{r},t)$ is the electron density, and ν is the collision frequency. Thus,

$$Q(\omega) = -\frac{1}{\sqrt{4\pi}\, c^2} \frac{e^2}{m\epsilon_o} \left(\frac{\omega}{\omega - i\nu} \right) (\hat{e}_R \cdot \hat{e}_T) \qquad (5.29)$$

and the channel function becomes

$$K(t,\omega) = -\frac{A_o}{\sqrt{4\pi}\,c^2}\,\frac{e^2}{m\epsilon_o}\left(\frac{\omega}{\omega - i\nu}\right)(\hat{e}_R \cdot \hat{e}_T)\, e^{-i\omega t_o}$$

$$\int_V G(\bar{r}')\, e^{i\frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot \bar{r}'}\, n(\bar{r}', t)\, dV' \qquad (5.30)$$

The channel function can be computed if the density profile is known. For example, let us consider a hypothetical model of the plasma for which we take to be a spherical column with a density

$$n(\bar{r}', t) = n_o e^{-\alpha(t)\frac{|\bar{r}'|^2}{a^2}} \quad ; \quad \alpha(t) > 0 \qquad (5.31)$$

where $\bar{r}'$ is a vector measured from the origin within the plasma column, a is a constant which has the dimension of length, and $\alpha(t)$ is a time-varying parameter which defines the instantaneous size of the plasma column. For simplicity we assume that the antenna illumination is uniform, $G(\bar{r}') \sim 1$, and that the plasma is collisionless, $\nu \sim 0$. With these assumptions, (5.30) becomes

$$K(t,\omega) = -\frac{A_o}{\sqrt{4\pi}}\,\frac{\omega_{op}^2}{c^2}\,(\hat{e}_R \cdot \hat{e}_T)\, e^{-i\omega t_o}$$

$$\iiint_{-\infty}^{+\infty} e^{-\frac{\alpha(t)}{a^2}|\bar{r}'(x',y',z')|^2}\, e^{i\frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot \bar{r}'(x',y',z')}\, dx'\, dy'\, dz'$$

$$(5.32)$$

where $\omega_{op}^2 = n_o e^2/m\epsilon_o$. In (5.32) we have assumed that $\alpha(t)/a^2$ is large enough so that $n(\overline{r}', t)$ is a rapidly decreasing function of $|\overline{r}'|$ and the integration may be extended from $-\infty$ to $+\infty$. Since

$$\overline{r}' = \hat{a}_x x' + \hat{a}_y y' + \hat{a}_z z'$$

and $\hat{a}_T$ and $\hat{a}_R$ are unit vectors from the origin to the transmitter and receiver, respectively, so that we have

$$\hat{a}_T = \hat{a}_x \sin\theta_T \cos\phi_T + \hat{a}_y \sin\theta_T \sin\phi_T + \hat{a}_z \cos\theta_T$$

$$\hat{a}_R = \hat{a}_x \sin\theta_R \cos\phi_R + \hat{a}_y \sin\theta_R \sin\phi_R + \hat{a}_z \cos\theta_R$$

Equation (5.32) may be rewritten as

$$K(t,\omega) = -\frac{A_o}{\sqrt{4\pi}}\,\frac{\omega_{op}^2}{c^2}\,(\hat{e}_R\cdot\hat{e}_T)\,e^{-i\omega t_o}$$

$$\iiint_{-\infty}^{+\infty} e^{-\frac{\sigma(t)}{a^2}(x'^2+y'^2+z'^2)}\, e^{i\frac{\omega}{c}(\eta_x x' + \eta_y y' + \eta_z z')}\, dx'\, dy'\, dz' \qquad (5.33)$$

where

$$\eta_x = (\sin\theta_T \cos\phi_T + \sin\theta_R \cos\phi_R)$$

$$\eta_y = (\sin\theta_T \sin\phi_T + \sin\theta_R \sin\phi_R)$$

$$\eta_z = (\cos\theta_T + \cos\theta_R)$$

Carrying out the integration in (5.33), we obtain

$$K(t,\omega) = \frac{A_o \pi a^3 \omega_{op}^2}{2c^2 \alpha^{3/2}(t)} (\hat{e}_R \cdot \hat{e}_T) \exp[-i\omega t_o]$$

$$\exp\left[-\frac{1}{4}\frac{\omega^2 a^2}{c^2 \alpha(t)}(\eta_x^2 + \eta_y^2 + \eta_z^2)\right] \qquad (5.34)$$

Noting that

$$\eta_x^2 + \eta_y^2 + \eta_z^2 = 4\cos^2\frac{\beta}{2} \qquad (5.35)$$

where β is the bistatic scattering angle,

$$\beta = \cos^{-1}(\hat{a}_T \cdot \hat{a}_R)$$

Equation (5.34) becomes

$$K(t,\omega) = -\frac{A_o \pi a^3 \omega_{op}^2}{2 c^2 \alpha^{3/2}(t)} (\hat{e}_R \cdot \hat{e}_T)$$

$$\exp\left[-i\omega t_o - \frac{a^2 \omega^2}{c^2 \alpha(t)} \cos^2\frac{\beta}{2}\right] \qquad (5.36)$$

From (4.36) the channel output is given by

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} K(t,\omega)F(\omega)\, e^{i\omega t}\, d\omega$$

Assume that the transmitted signal has a Gaussian envelope, i.e., $f(t) = (B/2\sqrt{\pi}) \exp[-B^2t^2/4]$, so that $F(\omega)$ takes the form

$$F(\omega) = \exp\left[-\omega^2/B^2\right] \tag{5.37}$$

Using (5.36) and (5.37), the channel output becomes

$$h(t) = -\frac{A_o\sqrt{\pi}\, a^3 \omega_{op}^2}{4\, c^2\, \alpha^{3/2}(t)} \frac{B}{\left(1 + \frac{a^2 B^2}{c^2 \alpha(t)} \cos^2\frac{\beta}{2}\right)^{1/2}} (\hat{e}_R \cdot \hat{e}_T)$$

$$\exp\left\{-\frac{1}{4} \frac{B^2}{\left(1 + \frac{a^2B^2}{c^2\alpha(t)}\cos^2\frac{\beta}{2}\right)} (t-t_o)^2\right\} \tag{5.38}$$

Thus, the transmitted pulse is widened and distorted as a result of the electron density fluctuations in the plasma column.

A special form of $\alpha(t)$ may be represented by

$$\alpha(t) = \alpha_o + \alpha_1 b(t)$$

where $|b(t)| \leq 1$ and $\alpha_1 b(t)$ fluctuates about the value α_o. If the time variation is small, $(\alpha_o/\alpha_1) << 1$, the channel output is approximately given by

$$h(t) \sim -\frac{A_o\sqrt{\pi}\, a^3 B\, \omega_{op}^2}{4\, \gamma c^2 \alpha_o^{3/2}} (\hat{e}_R \cdot \hat{e}_T)\left[1 - \left(\frac{\alpha_1}{\alpha_0}\right)\left(1 + \frac{1}{2\gamma^2}\right) b(t)\right]$$

$$\exp\left\{-\frac{1}{4}\frac{B^2}{\gamma^2}\left[1 + \left(\frac{\alpha_1}{\alpha_0}\right)\left(1 - \frac{1}{\gamma^2}\right) b(t)\right] (t-t_o)^2\right\} \tag{5.39}$$

where

$$\gamma^2 = 1 + \frac{a^2 B^2}{c^2 \alpha_o} \cos^2 \frac{\theta}{2}$$

In the stationary case, $b(t) = 0$, (5.38) reduces to

$$h(t) = -\frac{A_o \sqrt{\pi}\, a^3 B\, \omega_{op}^2}{4 \gamma c^2 \alpha_o^{3/2}} (\hat{e}_R \cdot \hat{e}_T) \exp\left\{-\frac{1}{4} \frac{B^2}{\gamma^2} (t-t_o)^2\right\} \tag{5.40}$$

5.4 Statistical Properties

Some of the statistical properties of the scatter channel are examined in this section. Again, we will restrict our analysis to that of a single constituent, described by a number density $n(\overline{r}, t)$. We assume that the medium is randomly varying so that $n(\overline{r}, t)$ becomes a random process in space and time.

From Chapter 4, we recall that the input-output relation for the channel can be written as

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K(t, \omega) F(\omega)\, e^{i\omega t}\, d\omega \tag{5.41}$$

where

$$K(t, \omega) = \int_{o}^{\infty} k(t, s)\, e^{-i\omega s}\, ds$$

and

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt$$

We assume that f(t) is non-random and k(t, s) is real so that $K(t,\omega) = K^*(t,-\omega)$. The mean of the channel output is given by

$$E\, h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\, K(t,\omega)\, F(\omega)\, e^{i\omega t}\, d\omega \qquad (5.42)$$

From (5.26), we have immediately

$$E\, K(t,\omega) = A_o Q(\omega)\, e^{-i\omega t_o} \int_V G(\bar{r}')\, e^{i\frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot \bar{r}'} < n(\bar{r}', t) > dV' \qquad (5.43)$$

where $< n(\bar{r}, t) > = E\, n(\bar{r}, t)$ is the mean number density.

The covariance of the channel output is given by

$$\text{Cov}[h(t), h(t')] = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \text{Cov}[K(t,\omega), K^*(t',\omega')]$$

$$\exp[i(\omega t - \omega' t')]\, F(\omega) F(\omega')\, d\omega\, d\omega' \qquad (5.44)$$

Using (5.26), we obtain

$$\text{Cov}[K(t,\omega), K^*(t',\omega')] = A_o^2 Q(\omega) Q^*(\omega') \exp[-i(\omega-\omega')t_o]$$

$$\int_V \int_V G(\bar{r}) G(\bar{r}') \exp\left[\frac{i}{c}(\hat{a}_T + \hat{a}_R)\cdot(\omega\bar{r} - \omega'\bar{r}')\right]$$

$$\text{Cov}[n(\bar{r}, t), n(\bar{r}', t')]\, dV\, dV' \qquad (5.45)$$

Thus, to compute the covariance of $h(t)$, we need the covariance of $n(\bar{r}, t)$. We suppose that the scattering is caused by inhomogeneities in the density of the medium in random flow. In order to facilitate the computations, we shall make certain simplifying assumptions regarding the flow. Let $\bar{t} = (t+t')/2$. We like to relate $n(\bar{r}, t)$ to $n(\bar{r}_1, \bar{t})$ and $n(\bar{r}', t')$ to $n(\bar{r}_2, \bar{t})$, where $\bar{r}_1$ is the point to which $\bar{r}$ flows as time goes from t to $\bar{t}$, and $\bar{r}_2$ flows into $\bar{r}'$ as time goes from $\bar{t}$ to t'. We assume that $t-t'$ is small and ignore effects, other than the flow, which can change the density of the constituent being considered. Then, by means of an equation of continuity, we have

$$\begin{aligned} n(\bar{r}_1, \bar{t})\, dV_1 &= n(\bar{r}, t)\, dV \\ n(\bar{r}_2, \bar{t})\, dV_2 &= n(\bar{r}', t')\, dV' \end{aligned} \tag{5.46}$$

where dV_1 and dV_2 are two elementary volumes at $\bar{r}_1$ and $\bar{r}_2$, respectively. We also assume that the velocity of the random flow is independent of the density $n(\bar{r}, t)$. Then, for small values of $t-t'$, we may write

$$\begin{aligned} \bar{r}_1 &= \bar{r} + \bar{v}(\bar{r}_1, \bar{t})\left(\frac{t'-t}{2}\right) \\ \bar{r}_2 &= \bar{r}' - \bar{v}(\bar{r}_2, \bar{t})\left(\frac{t'-t}{2}\right) \end{aligned} \tag{5.47}$$

where $\bar{v}(\bar{r}, t)$ is the velocity of the flow at $(\bar{r}, t)$. Thus, if $\phi(\bar{r})$ is some function of position and if

$$\varphi = \int_V \phi(\bar{r})\, n(\bar{r}, t)\, dV$$

$$\varphi' = \int_V \phi'(\bar{r}')\, n(\bar{r}', t')\, dV'$$

then, from (5.46) and (5.47), φ and φ' can also be expressed as

$$\varphi = \int_{V_1} \phi\left[\bar{r}_1 - \bar{v}(\bar{r}_1, \bar{t})\left(\frac{t'-t}{2}\right)\right] n(\bar{r}_1, \bar{t})\, dV_1$$

$$\varphi' = \int_{V_2} \phi'\left[\bar{r}_2 + \bar{v}(\bar{r}_2, \bar{t})\left(\frac{t'-t}{2}\right)\right] n(\bar{r}_2, \bar{t})\, dV_2$$

It follows that

$$\mathrm{Cov}\,[\varphi, \varphi'] = \int_V \int_{V'} \phi(\bar{r})\phi'(\bar{r}')\, \mathrm{Cov}\,[n(\bar{r}, t), n(\bar{r}', t')]\, dV\, dV'$$

$$= \int_{V_1}\int_{V_2} E\, \phi\left[\bar{r}_1 - \bar{v}(\bar{r}_1, \bar{t})\left(\frac{t'-t}{2}\right)\right]\phi'\left[\bar{r}_2 + \bar{v}(\bar{r}_2, \bar{t})\left(\frac{t'-t}{2}\right)\right]$$

$$\mathrm{Cov}\,[n(\bar{r}_1, \bar{t}), n(\bar{r}_2, \bar{t})]\, dV_1 dV_2 \qquad (5.48)$$

In this manner, we may put

$$\mathrm{Cov}\,[n(\bar{r}_1, \bar{t}), n(\bar{r}_2, \bar{t})] = A\left(\frac{\bar{r}_1 + \bar{r}_2}{2}, \bar{r}_2 - \bar{r}_1, \bar{t}\right) \qquad (5.49)$$

Applying (5.48) to (5.45), we obtain

$$\mathrm{Cov}\left[K(t,\omega), K^*(t';\omega')\right] = A_o^2 Q(\omega) Q^*(\omega') \exp\left[-i(\omega-\omega')t_o\right]$$

$$\int_{V_1}\int_{V_2} \exp\left[-\frac{i}{c}(\hat{a}_T+\hat{a}_R)\cdot(\omega'\bar{r}_2-\omega\bar{r}_1)\right]$$

$$E\, G\left[\bar{r}_1-\bar{v}(\bar{r}_1,\bar{t})\left(\frac{t'-t}{2}\right)\right] G^*\left[\bar{r}_2+\bar{v}(\bar{r}_2,\bar{t})\left(\frac{t'-t}{2}\right)\right]$$

$$\exp\left\{-\frac{i}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[\omega'\bar{v}(\bar{r}_2,\bar{t})+\omega\bar{v}(\bar{r}_1,\bar{t})\right]\left(\frac{t'-t}{2}\right)\right\}$$

$$A\left(\frac{\bar{r}_1+\bar{r}_2}{2}, \bar{r}_2-\bar{r}_1, \bar{t}\right) dV_1\, dV_2 \qquad (5.50)$$

If we define

$$\bar{r} = \frac{\bar{r}_1+\bar{r}_2}{2} \qquad \text{and} \qquad \bar{\rho} = \bar{r}_2-\bar{r}_1$$

Equation (5.50) may be written as

$$\mathrm{Cov}\left[K(t,\omega), K^*(t',\omega')\right] = A_o^2 Q(\omega) Q^*(\omega') \exp\left[-i(\omega-\omega')t_o\right]$$

$$\iint \exp\left\{\frac{i}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[(\omega-\omega')\bar{r}-(\omega+\omega')\frac{\bar{\rho}}{2}\right]\right.$$

$$E\, G\left[\bar{r}-\frac{\bar{\rho}}{2}-\bar{v}\left(\bar{r}-\frac{\bar{\rho}}{2},\bar{t}\right)\left(\frac{t'-t}{2}\right)\right] G^*\left[\bar{r}+\frac{\bar{\rho}}{2}+\bar{v}\left(\bar{r}+\frac{\bar{\rho}}{2},\bar{t}\right)\left(\frac{t'-t}{2}\right)\right]$$

$$\exp\left\{-\frac{i}{c}\left(\frac{\omega'-\omega}{2}\right)\left(\frac{t'-t}{2}\right)(\hat{a}_T+\hat{a}_R)\cdot\left[\bar{v}\left(\bar{r}+\frac{\bar{\rho}}{2},\bar{t}\right)-\bar{v}\left(\bar{r}-\frac{\bar{\rho}}{2},\bar{t}\right)\right]\right\}$$

$$\exp\left\{-\frac{i}{c}\left(\frac{\omega'+\omega}{2}\right)\left(\frac{t'-t}{2}\right)(\hat{a}_T+\hat{a}_R)\cdot\left[\bar{v}\left(\bar{r}+\frac{\bar{\rho}}{2},\bar{t}\right)+\bar{v}\left(\bar{r}-\frac{\bar{\rho}}{2},\bar{t}\right)\right]\right\}$$

$$A(\bar{r},\bar{\rho},\bar{t})\, d^3\bar{r}\, d^3\bar{\rho} \qquad (5.51)$$

We assume that $A(\bar{r},\bar{\rho},\bar{t})$ varies slowly with r and vanishes relatively rapidly as $|\bar{\rho}|$ increases. Specifically, we assume that $A(\bar{r},\bar{\rho},\bar{t})$ becomes negligibly small when $|\bar{\rho}|$ exceeds some correlation length ρ_c which is small compared to the modulation wavelength $\lambda_m = 2\pi c/B$, where B is the signal bandwidth. This implies that the signal illuminates many uncorrelated scattering volumes with nearly uniform modulation. Thus, by keeping ρ sufficiently small, we may assume that

$$\bar{v}(\bar{r} \pm \frac{\bar{\rho}}{2}, \bar{t}) \approx \bar{v}(\bar{r}, \bar{t})$$

Equation (5.51) becomes

$$\begin{aligned}
&\mathrm{Cov}\left[K(t,\omega), K^*(t',\omega')\right] = A_o^2 Q(\omega)Q^*(\omega') \exp\left[-i(\omega-\omega')t_o\right] \\
&\iiint \exp\left\{+\frac{i}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[(\omega-\omega')\bar{r} - (\omega+\omega')\frac{\bar{\rho}}{2}\right]\right\} \\
&\qquad G\left[\bar{r}-\frac{\bar{\rho}}{2}-\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right]G^*\left[\bar{r}+\frac{\bar{\rho}}{2}+\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right] \\
&\qquad \exp\left[-\frac{i}{c}(\omega+\omega')\left(\frac{t'-t}{2}\right)(\hat{a}_T+\hat{a}_R)\cdot\bar{v}(\bar{r},\bar{t})\right] \\
&\qquad F(\bar{v};\bar{r};\bar{t})\, A(\bar{r},\bar{\rho},\bar{t})\, d^3\bar{v}\, d^3\bar{r}\, d^3\bar{\rho}
\end{aligned} \tag{5.52}$$

where $F(\bar{v};\bar{r},t)$ is the probability density for the velocity of the flow at $(\bar{r},t)$.

Now we assume that $Q(\omega)$ is wideband, i.e., $Q(\omega) \approx Q(\omega_o)$, and

that $\frac{v}{c}(t'-t)B << 1$, where ω_o is the angular carrier frequency and B is the signal bandwidth. Substitute (5.52) into (5.44), we obtain

$$\mathrm{Cov}\,[h(t),h(t')] = \frac{A_o^2}{(2\pi)^2}\,|Q(\omega_o)|^2$$

$$\iiint f\left\{t-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[\bar{r}-\frac{\bar{\rho}}{2}-\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right]\right\}$$

$$f\left\{t'-t_o+\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[\bar{r}+\frac{\bar{\rho}}{2}+\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right]\right\}$$

$$G\left[\bar{r}-\frac{\bar{\rho}}{2}-\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right]G^*\left[\bar{r}+\frac{\bar{\rho}}{2}+\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right]$$

$$F(\bar{v};\bar{r},\bar{t})\,A(\bar{r},\bar{\rho},\bar{t})\,d^3\bar{v}\,d^3\bar{r}\,d^3\bar{\rho} \qquad (5.53)$$

where

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\,e^{i\omega t}\,d\omega$$

As in the discrete scatterer case, we assume that $f(t)$ is of the form,

$$f(t) = m(t)\cos\left[\omega_o t + \varphi_o(t)\right] \qquad (5.54)$$

where $m(t)$ and $\varphi_o(t)$ are slowly varying functions of time. Substitute (5.54) in (5.53) and assume that ρ and $t'-t$ are sufficiently small, so that we may drop the quantities

$$\pm\frac{1}{c}(\hat{a}_T+\hat{a}_R)\cdot\left[\frac{\bar{\rho}}{2}+\bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right]$$

from the arguments of $m(t)$ and $\omega_o(t)$. We then use the sum and difference identity on the product of cosines and drop the sum term because of its rapid variations with r. Equation (5.53) becomes

$$\text{Cov}\,[\,h(t),h(t')\,] = \frac{A_o^2}{2}\,|Q(\omega_o)|^2 \iiint F(\bar{v};\bar{r},\bar{t})\,A(\bar{r},\bar{\rho},\bar{t})$$

$$G\left[\,\bar{r} - \frac{\bar{\rho}}{2} - \bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right] G^*\left[\,\bar{r} + \frac{\bar{\rho}}{2} + \bar{v}(\bar{r},\bar{t})\left(\frac{t'-t}{2}\right)\right]$$

$$m[\,t-t_o + \frac{1}{c}\,(\hat{a}_T + \hat{a}_R)\cdot\bar{r}\,]\,m[\,t'-t_o + \frac{1}{c}\,(\hat{a}_T + \hat{a}_R)\cdot\bar{r}\,]$$

$$\cos\left\{\frac{\omega_o}{c}\,(\hat{a}_T + \hat{a}_R)\cdot[\,\bar{\rho} + \bar{v}(\bar{r},\bar{t})(t'-t)\,]\right\} d^3\bar{v}\, d^3\bar{r}\, d^3\bar{\rho} \qquad (5.55)$$

The antenna gain patterns can influence the covariance function greatly when the value of $\bar{v}(\bar{r},\bar{t})(t'-t)$ exceeds the correlation length ρ_c. As $\bar{v}(\bar{r},\bar{t})(t'-t)$ increases to a large enough value, the integration will be carried over disjoint volumes of the scattering material, and the covariance function will drop rapidly to zero.

The variance of the channel output can be obtained directly from (5.55). Thus

$$\text{Var } h(t) = \text{Cov}\,[h(t), h(t)]$$

$$= \frac{A_o^2}{2}\,|Q(\omega_o)|^2 \iint A(\bar{r}, \bar{\rho}, t)\,G(\bar{r} - \frac{\bar{\rho}}{2})\,G^*(\bar{r} + \frac{\bar{\rho}}{2})$$

$$m^2[t - t_o + \frac{1}{c}(\hat{a}_T + \hat{a}_R)\cdot\bar{r}]\cos\left[\frac{\omega_o}{c}(\hat{a}_T + \hat{a}_R)\cdot\bar{\rho}\right] d^3\bar{r}\, d^3\bar{\rho} \tag{5.56}$$

Since $\frac{1}{2}m^2(t)$ is proportional to the average power being transmitted at t, hence Var h(t) is proportional to the average power in the fluctuating component of h(t), received at t. Substitute the value of A_o^2 from (5.22) in (5.56) and compare with the bistatic radar equation, we see that the channel cross section is given by

$$\sigma(t) = |Q(\omega_o)|^2 \iint A(\bar{r}, \bar{\rho}, t)\,G(\bar{r} - \frac{\bar{\rho}}{2})\,G^*(\bar{r} + \frac{\bar{\rho}}{2})$$

$$\cos\left[\frac{\omega_o}{c}(\hat{a}_T + \hat{a}_R)\cdot\bar{\rho}\right] d^3\bar{r}\, d^3\bar{\rho} \tag{5.57}$$

In many practical situations, it is possible to put

$$A(\bar{r}, \bar{\rho}, t) = C(\bar{\rho})\,\text{Var } n(\bar{r}, t)$$

where C(0) = 1. Then (5.57) may be written as

$$\sigma(t) = |Q(\omega_o)|^2 \iint G(\bar{r} - \frac{\bar{\rho}}{2})\,G^*(\bar{r} + \frac{\bar{\rho}}{2})\,\text{Var } n(\bar{r}, t)$$

$$C(\bar{\rho})\cos\left[\frac{\omega_o}{c}(\hat{a}_T + \hat{a}_R)\cdot\rho\right] d^3\bar{r}\, d^3\bar{\rho} \tag{5.58}$$

If the antenna gains are nearly uniform, (5.58) reduces to

$$\sigma(t) = |Q(\omega_o)|^2 V_c(\omega_o) \int_V \text{Var } n(\bar{r}, t)\, d^3\bar{r} \tag{5.59}$$

where $V_c(\omega)$ is the correlation volume,

$$V_c(\omega) = \int C(\bar{\rho}) \cos\left[\frac{\omega}{c}(\hat{a}_T + \hat{a}_R)\cdot\bar{\rho}\right] d^3\bar{\rho} \tag{5.60}$$

For the plasma scattering example of the last section, the quantity $|Q(\omega_o)|^2$ is given by

$$|Q(\omega_o)|^2 = \frac{1}{4\pi c^2}\left(\frac{e^2}{m\epsilon_o}\right)^2\left(\frac{\omega_o^2}{\omega_o^2+\nu^2}\right)|(\hat{e}_T\cdot\hat{e}_R)|^2 . \tag{5.61}$$

The evaluation of $V_c(\omega)$ depends upon the particular model used for the correlation function $C(\bar{\rho})$. A possible form of $C(\bar{\rho})$ frequently assumed is

$$C(\bar{\rho}) = e^{-|\bar{\rho}|^2/\rho_c^2} \tag{5.62}$$

Substitute (5.62) in (5.60) and carry out the integration using the same method as that for evaluating (5.32), we obtain

$$V_c(\omega) = \pi^{3/2}\rho_c^3\, e^{-\frac{\omega^2}{c^2}\rho_c^2\cos^2\frac{\beta}{2}} \tag{5.63}$$

where β is the bistatic angle.

Appendix I

APPLICATION OF THE LORENTZ RECIPROCITY THEOREM TO SCATTERING PROBLEMS

Let $(\bar{E}_1, \bar{H}_1)$ and $(\bar{E}_2, \bar{H}_2)$ be two different distributions of electromagnetic fields of the same frequency, which can be established in a region of space containing no sources. Each set of field vectors satisfies the Maxwell's equations. Lorentz reciprocity theorem st...

$$\int_S \hat{n} \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1)\, dS = 0 \qquad \text{(I.1)}$$

where $\hat{n}$ is the unit normal vector to the surface S enclosing the region of interest. Although there are many forms of the reciprocity theorem [32] - [35], the form we use here is that due to Lorentz [13], [36].

In general, the scattering geometry consists of a transmitter which illuminates the target and a receiver at which we observe the scattered signal. A schematic representation is shown in Fig. 8. In particular, we seek an expression for the voltage induced at the receiver terminals by the field scattered by the target.

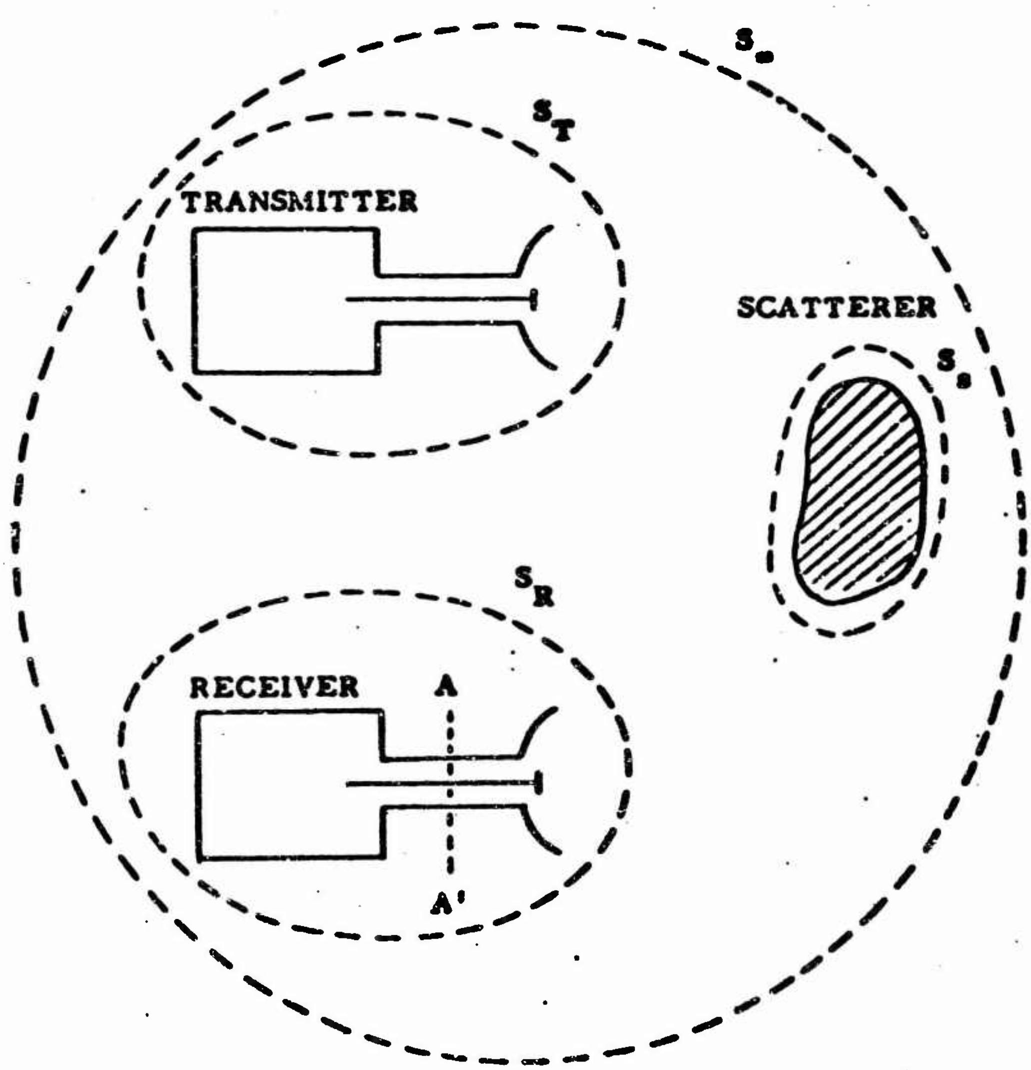

Figure 8. Schematic Representation of Scattering Geometry

Let $(\bar{E}_1, \bar{H}_1)$ be the fields of the receiving antenna with driving current I_R at its feed point with no other sources present; i.e., both the transmitter source and target are removed. Let $(\bar{E}_2, \bar{H}_2)$ be the total fields obtained with both the transmitter source and the target in place. It is assumed that the same current I_R is delivered to the feed point of the receiving antenna either with the transmitter source and the target removed or in place. Thus, $\bar{E}_1$ and $\bar{H}_1$ represent the total fields which would be obtained in space if the receiving antenna were used as a transmitting antenna, while $\bar{E}_2$ and $\bar{H}_2$ represent the sum of the transmitter, receiver, and scattered fields.

The surface integral in (I.1) can be written as

$$\int_{S_\infty + S_T + S_s + S_R} \hat{n} \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1)\, dS = 0 \qquad \text{(I.2)}$$

The integration over the spherical surface S_∞ vanishes on account of the radiation condition as the sphere recedes to infinity [37]. For simplicity, we replace the transmitter by an equivalent source $\bar{J}_T$, which creates the incident field. From the divergence theorem, we have

$$\int_{S_T} \hat{n} \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) dS = -\int_{V_T} \nabla \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) dV \qquad \text{(I.3)}$$

Within the volume V_T, Maxwell's equations are

$$\nabla \times \bar{E}_1 = -i\omega\mu_o\bar{H}_1 \qquad \nabla \times \bar{E}_2 = -i\omega\mu_o\bar{H}_2$$

$$\nabla \times \bar{H}_1 = +i\omega\epsilon_o\bar{E}_1 \qquad \nabla \times \bar{H}_2 = +i\omega\epsilon_o\bar{E}_2 + \bar{J}_T$$

Eq. (I.3) becomes

$$\int_{S_T} \hat{n}\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)dS = \int_{V_T} \bar{E}_1\cdot\bar{J}_T \qquad (I.4)$$

The integral over the surface of the scatterer can be evaluated in a similar fashion. That is, we replace the scatterer by an equivalent distribution of current $\bar{J}_s$, induced by the incident field. Then, within the volume V_s, enclosed by the surface S_s, the fields satisfy Maxwell's equations

$$\nabla \times \bar{E}_1 = -i\omega\mu_o\bar{H}_1 \qquad \nabla \times \bar{E}_2 = -i\omega\mu_o\bar{H}_2$$

$$\nabla \times \bar{H}_1 = +i\omega\epsilon_o\bar{H}_1 \qquad \nabla \times \bar{H}_2 = +i\omega\epsilon_o\bar{E}_2 + \bar{J}_s$$

Again, applying the divergence theorem, we obtain

$$\int_{S_s} \hat{n}\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)dS = \int_{V_s} \bar{E}_1\cdot\bar{J}_s dV \qquad (I.5)$$

The surface S_R includes the exterior shielding surfaces (assumed to be perfectly conducting) of the receiving system, the inner metal surfaces of the antenna and transmission line down to the

feed point (shown by the plane AA' in Fig. 8), and the cross-sectional plane AA' inside the transmission line. Since $\hat{n}\cdot\bar{E}_1\times\bar{H}_2 = \hat{n}\times\bar{E}_1\cdot\bar{H}_2 = 0$ and $\hat{n}\cdot\bar{E}_2\times\bar{H}_1 = \hat{n}\times\bar{E}_2\cdot\bar{H}_1 = 0$ on the metal surfaces, the only non-vanishing contribution to the surface integral over S_R is that over the cross-sectional plane AA'. Thus,

$$\int_{S_R} \hat{n}\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)dS = \int_{AA'} \hat{n}\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)dS \qquad (I.6)$$

The evaluation of the surface integral over AA' requires some consideration of the field distribution inside the transmission line. For the present purpose we consider a coaxial line [13], although such a choice is arbitrary [36]. For a coaxial line, the fields in cylindrical coordinates are given by

$$E_r = \frac{V}{r\ ln(b_o/b_i)}\ , \qquad H_\phi = \frac{I}{2\pi r} \qquad (I.7)$$

where b_i and b_o are the inner and outer radii of the line, and V and I are the line voltage and current. Substitution of (I.7) into (I.6) gives

$$\int_{AA'} \hat{n}\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1)dS = V_1 I_2 - V_2 I_1 \qquad (I.8)$$

Let V' be the total voltage induced at the terminals of the receiver by the transmitter field and the scattered field, then $V_2 = V_1 + V'$,

$I_2 = (V_1 - V')/Z_o$, and $V_1 I_2 - V_2 I_1 = 2I_1 V'$, where Z_o is the characteristic impedance of the transmission line. Thus, we obtain

$$\int_{S_R} \hat{n} \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) dS = -2I_1 V' \qquad (I.9)$$

Finally, from Equations (I.2),(I.4), (I.5), and (I.9), we have

$$V' = \frac{1}{2I_1} \left[\int_{V_T} \bar{E}_1 \cdot \bar{J}_T dV + \int_{V_s} \bar{E}_1 \cdot \bar{J}_s dV \right] \qquad (I.10)$$

The first term is due to transmitter-receiver coupling. Hence, the scattered voltage is given by

$$\tilde{a} = \frac{1}{2I_1} \int_{V_s} \bar{E}_1 \cdot \bar{J}_s dV \qquad (I.11)$$

Appendix II

PROBABILITY DENSITY FUNCTION FOR THE RETURNED WAVEFORM FROM A RANDOM COLLECTION OF ROTATING DIPOLE SCATTERERS

The probability density function for the echo waveform can be evaluated by first finding the characteristic function of the distribution [24], [25]. For simplicity we let

$$\int \cdots \int_{w_i} p(\vec{a}_0, \vec{\theta}, \vec{\phi}, \vec{\xi}, \vec{\eta}, \vec{\alpha}, \vec{\beta}, \vec{\omega}_d, \vec{\omega}_r) d\vec{a}_0 d\vec{\theta} \ldots d\vec{\omega}_r$$
$$= \int_{-\infty}^{+\infty} \cdots \int p(\vec{a}_0) d\vec{a}_0 \int_0^{\pi} \cdots \int p(\vec{\theta}) d\vec{\theta} \ldots \int_{-\infty}^{+\infty} \cdots \int p(\vec{\omega}_r) d\vec{\omega}_r$$

and

$$p(\vec{w}) d\vec{w} = \prod_{i=1}^{n} p(w_i)\, dw_i$$

where $\vec{w}$ represents $\vec{a}_0, \vec{\theta}, \vec{\phi}, \ldots$ etc. Hence, we can write,

$$p(x) = \sum_{n=0}^{\infty} \int \cdots \int_{w_i} p(\vec{a}_0, \ldots, \vec{\omega}_r) d\vec{a}_0 \ldots d\vec{\omega}_r \int_{-T}^{+T} \cdots \int p(\vec{t}) d\vec{t}\, \delta(x - x_n) \qquad \text{(II. 1)}$$

From Eqs. (3.2) and (3.4), the echo signal can be written in the form,

$$x(t) = \sum_{k=1}^{n} \Phi_k(t, t_k)\left[C \cos \phi_1 + U \cos(\phi_1 + \phi_s) + L \cos(\phi_1 - \phi_s)\right] \tag{II.2}$$

where

$$\Phi_k(t, t_k) = \frac{K_1}{r_0^2} G(\theta_k, \phi_k) a_{0k} s(t - t_k)$$

$$\phi_1 = (\omega_0 + \omega_{dk})(t - t_k) + \omega_0(t - t_k) + \beta_k + \epsilon(\theta_k, \phi_k) + \chi(\xi_k, \eta_k)$$

$$\phi_s = 2\omega_{rk}\left(t - \frac{t_k}{2}\right) + 2\alpha_k + \gamma_k(\xi_k, \eta_k)$$

The characteristic function of the distribution becomes

$$\Phi(u) = \int_{-\infty}^{+\infty} p(x)\, e^{iux}\, dx$$

$$= \sum_{n=0}^{\infty} \int \cdots \int_{w_1} p(\vec{a}_0, \ldots, \vec{\omega}_r) d\vec{a}_0 \ldots d\vec{\omega}_r \int_{-T}^{+T} \cdots \int \prod_{k=1}^{n} \frac{\nu(t_k)}{k} e^{-\int_{-T}^{+T} \nu(t)dt}\, dt_k$$

$$\cdot \exp iu \sum_{k=1}^{n} \Phi_k(t, t_k)\left[C \cos \phi_1 + U \cos(\phi_1 + \phi_s) + L \cos(\phi_1 - \phi_s)\right]$$

$$= e^{-\int_{-T}^{+T} \nu(t)dt} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \int_{-T}^{+T} \nu(t')dt' \int \cdots \int_{w_1} p(\vec{a}_0 \ldots \vec{\omega}_r) d\vec{a}_0 \ldots d\vec{\omega}_r \right.$$

$$\left. \cdot \exp\left[iu\, \Phi_k(t, t')(C \cos \phi_1 + U \cos(\phi_1 + \phi_s) + L \cos(\phi_1 - \phi_s))\right]\right\}^n$$

$$= \exp\left\{ -\int_{-T}^{+T} \nu(t)dt + \int_{-T}^{+T} \nu(t')dt' \int \cdots \int_{w_1} p(\vec{a}_0, \ldots, \vec{\omega}_r) d\vec{a}_0 \ldots d\vec{\omega}_r \right.$$

$$\left. \cdot \exp\left[iu\, \Phi_k(t, t')(C \cos \phi_1 + U \cos(\phi_1 + \phi_s) + L \cos(\phi_1 - \phi_s))\right]\right\}$$

Expanding the exponential insid: the integral in a power series, we obtain

$$\phi(u) = \exp\left\{-\int_{-T}^{+T}\nu(t)dt + \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{-T}^{+T}\nu(t')dt'\int\cdots\int_{w_i} p(\vec{a}_0,\ldots,\vec{\omega}_r)\right.$$

$$d\vec{a}_0\ldots d\vec{\omega}_r\left[iu\,\phi_k(t,t')\right]^{\ell+m+n}\frac{C^\ell U^m L^n}{\ell!\,m!\,n!}\cos^\ell\phi_1\cos^m(\phi_1+\phi_s)$$

$$\cos^n(\phi_1-\phi_s) = \exp\left\{-\frac{u^2}{2}\sigma^2 + \int_{-T}^{+T}\nu(t')dt'\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\right.$$

$$\left.\frac{(iu)^{\ell+m+n}}{\ell!\,m!\,n!}W(\ell,m,n;t,t')\right\}$$

where, for simplicity, we have set $\langle x\rangle = 0$, $\langle x^2\rangle = \sigma^2$, and

$$W(\ell,m,n;t,t') = \int\cdots\int_{w_i} p(\vec{a}_0,\ldots,\vec{\omega}_r)d\vec{a}_0\ldots d\vec{\omega}_r\left[\phi_k(t,t')\right]^{\ell+m+n}$$

$$\cdot\, C^\ell U^m L^n\cos^\ell\phi_1\cos^m(\phi_1+\phi_s)\cos^n(\phi_1-\phi_s)$$

More conveniently, we can write the triple summation as

$$\sum_{q=e}^{\infty}\frac{(iu)^q}{q!}W_q(t,t') \qquad \text{where } q = \ell+m+n$$

Thus,

$$\phi(u) = \exp\left\{-\frac{u^2}{2}\sigma^2 + \int_{-T}^{+T}\nu(t')dt'\sum_{q=3}^{\infty}\frac{(iu)^q}{q!}W_q(t,t')\right\} \qquad \text{(III.3)}$$

By taking the inverse transform of the characteristic function, we obtain

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(u)\, e^{-iux}\, du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[-iux - \frac{u^2}{2}\sigma^2\right] \cdot \exp\left[\sum_{q=3}^{\infty} \frac{(iu)^q}{q!}\sigma_q(t)\right] du \qquad (II.4)$$

where

$$\sigma_q(t) = \int_{-T}^{+T} \nu(t')W_q(t,t')dt' \qquad (II.5)$$

If we expand the second exponential of (II.4) in a power series, we have

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux - \frac{u^2}{2}\sigma^2}\left\{1 + \frac{(iu)^3}{3!}\sigma_3 + \frac{(iu)^4}{4!}\sigma_4 + \cdots\right\} du \qquad (II.6)$$

Equation (II.6) gives the time-varying probability distribution of the echo waveform. We will show that the probability density at a particular instant of time tends to become a Gaussian probability density as the echo rate becomes large. If we let

$$\alpha = \frac{x}{\sigma} \quad \text{and} \quad \gamma = \sigma u$$

Equation (II.6) can be written as

$$p(\alpha) = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty} e^{-i\gamma\alpha - \frac{\gamma^2}{2}}\left\{1 + \frac{(i\gamma)^3}{3!}\frac{\sigma_3}{\sigma^3} + \frac{(i\gamma)^4}{4!}\frac{\sigma_4}{\sigma^4} + \cdots\right\} d\gamma$$

We note that

$$\int_{-\infty}^{+\infty} (i\gamma)^n e^{-i\gamma\alpha - \frac{\gamma^2}{2}} d\gamma = (-1)^n \frac{d^n}{d\alpha^n}\int_{-\infty}^{+\infty} e^{-i\gamma\alpha - \frac{\gamma^2}{2}} d\gamma$$

Thus, evaluating the integrals and putting $p(x) = \sigma p(\alpha)$, we obtain

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}\left[1 + \sum_{k=3}^{\infty}\frac{1}{k!}\frac{\sigma_k}{\sigma^k} H_k\left(\frac{x}{\sigma}\right)\right] \tag{II.7}$$

where H_k are Hermite polynomials defined as [38],

$$H_k(\alpha) = (-1)^k e^{\frac{\alpha^2}{2}} \frac{d^k}{d\alpha^k} e^{-\frac{\alpha^2}{2}} .$$

We assume here that all moments of the distribution exist and note that σ^k can be written as

$$\sigma^k = \left[\int_{-T}^{+T} \nu(t')W_2(t,t')dt'\right]^{\frac{k}{2}} \tag{II.8}$$

From Eqs. (II-5) and (II.8), we have

$$\frac{\sigma_k}{\sigma^k} = K \frac{\int_{-T}^{+T} \nu(t') W_k(t,t') dt'}{\left[\int_{-T}^{+T} \nu(t') W_2(t,t') dt'\right]^{\frac{k}{2}}} \quad ; \quad K = \text{constant}$$

Now if we let ν_0 be the upper limit of the echo rate such that $\nu(t) \leq \nu_0$, then

$$\frac{\int_{-T}^{+T} \nu(t') W_k(t,t') dt'}{\left[\int_{-T}^{+T} \nu(t') W_2(t,t') dt'\right]^{\frac{k}{2}}} \leq \frac{\nu_0}{\nu_0^{k/2}} \frac{\int_{-T}^{+T} W_k(t,t') dt'}{\left[\int_{-T}^{+T} \frac{\nu(t')}{\nu_0} W_2(t,t') dt'\right]^{\frac{k}{2}}}$$

$$\leq \frac{1}{\nu_0^{k/2-1}} \frac{\int_{-T}^{+T} W_k(t,t') dt'}{\left[\int_{-T}^{+T} W_2(t,t') dt'\right]^{\frac{k}{2}}}$$

$$\to 0 \text{ as } \nu_0 \to \infty$$

Thus, as ν_0 becomes large, Eq. (II. 7) reduces to

$$p(x) \approx \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} \tag{II. 9}$$

The error is of order $\frac{1}{\nu^{k/2-1}}$ where $k \geq 3$.

In a similar manner, it can be shown that all joint distributions of the echo waveform approach Gaussian in the limit.

Appendix III

DYADIC GREEN'S FUNCTION FOR THE VECTOR HELMHOLTZ EQUATION

The vector Helmholtz equation for the electric field with $e^{-i\omega t}$ time dependence is

$$\nabla \times \nabla \times \bar{E}(\bar{r}) - k^2 \bar{E}(\bar{r}) = -i\omega\mu \bar{J}(\bar{r}) \tag{III.1}$$

where $k^2 = \omega^2 \mu\epsilon$. A solution for the inhomogeneous vector wave equation can be written as [28],

$$\bar{E}(\bar{r}) = -i\omega\mu \int_V \bar{\bar{\Gamma}}(\bar{r},\bar{r}') \cdot \bar{J}(\bar{r}')dV' \tag{III.2}$$

where the double bar indicates a dyadic quantity. Substitution of (III.2) into (III.1) gives

$$\int_V [\nabla \times \nabla \times \bar{\bar{\Gamma}}(\bar{r},\bar{r}') - k^2 \bar{\bar{\Gamma}}(\bar{r},\bar{r}')] \cdot \bar{J}(\bar{r}')dV' = \bar{J}(\bar{r}) \tag{III.3}$$

Since

$$\bar{J}(\bar{r}) = \int_V \bar{\bar{I}} \cdot \bar{J}(\bar{r}')\delta(\bar{r}-\bar{r}')dV'$$

where $\bar{\bar{I}}$ is the identity dyadic. Equation (III.3) becomes

$$\int_V [\nabla \times \nabla \times \overline{\overline{\Gamma}}(\overline{r},\overline{r}') - k^2 \overline{\overline{\Gamma}}(\overline{r},\overline{r}') - \overline{\overline{I}}\,\delta(\overline{r}-\overline{r}')] \cdot \overline{J}(\overline{r}')dV' = 0$$

Since this relation holds for any arbitrary $\overline{J}(\overline{r}')$, $\overline{\overline{\Gamma}}(\overline{r},\overline{r}')$ must satisfy the differential equation,

$$\nabla \times \nabla \times \overline{\overline{\Gamma}}(\overline{r},\overline{r}') - k^2 \overline{\overline{\Gamma}}(\overline{r},\overline{r}') = \overline{\overline{I}}\,\delta(\overline{r}-\overline{r}') \qquad \text{(III.4)}$$

We require that the solution satisfies the radiation condition for $r \to \infty$. The r-dependence must be in the form of an outgoing spherical wave.

Using the vector identity $\nabla \times \nabla \times \overline{A} = \nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A}$, (III.4) may be written as

$$(\nabla^2 + k^2)\overline{\overline{\Gamma}}(\overline{r},\overline{r}') = \nabla[\nabla \cdot \overline{\overline{\Gamma}}(\overline{r},\overline{r}')] - \overline{\overline{I}}\,\delta(\overline{r}-\overline{r}') \qquad \text{(III.5)}$$

Taking the divergence of (III.4), we find

$$\nabla \cdot \overline{\overline{\Gamma}}(\overline{r},\overline{r}') = -\frac{1}{k^2} \nabla\,\delta(\overline{r}-\overline{r}')$$

Equation (III.5) becomes

$$(\nabla^2 + k^2)\overline{\overline{\Gamma}}(\overline{r},\overline{r}') = -\left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right)\delta(\overline{r}-\overline{r}') \qquad \text{(III.6)}$$

Let

$$\overline{\overline{\Gamma}}(\overline{r},\overline{r}') = \left(\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla\right) G(\overline{r},\overline{r}') \qquad \text{(III.7)}$$

and note that the scalar operator $(\nabla^2 + k^2)$ on the left side of (III.6) may be transposed with the dyadic $(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla)$. We obtain

$$(\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla)[(\nabla^2 + k^2)G(\bar{r},\bar{r}') + \delta(\bar{r}-\bar{r}')] = 0 \qquad \text{(III.8)}$$

Equation (III.8) is satisfied by

$$(\nabla^2 + k^2)G(\bar{r},\bar{r}') = -\delta(\bar{r}-\bar{r}') \qquad \text{(III.9)}$$

which is the differential equation for the scalar Green's function. The solution in free space [30] is

$$G(\bar{r},\bar{r}') = \frac{e^{-ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \qquad \text{(III.10)}$$

Equations (III.7) and (III.10) yield the desired dyadic Green's function in free space

$$\bar{\bar{\Gamma}}(\bar{r},\bar{r}') = (\bar{\bar{I}} + \frac{1}{k^2}\nabla\nabla)\frac{e^{-ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \qquad \text{(III.11)}$$

To find a complete solution to the inhomogeneous vector wave equation, consider the vector form of the Green's theorem [37],

$$\int_V (\bar{Q}\cdot\nabla\times\nabla\times\bar{P} - \bar{P}\cdot\nabla\times\nabla\times\bar{Q})\,dV$$

$$= -\int_S (\bar{P}\times\nabla\times\bar{Q} - Q\times\nabla\times\bar{P})\cdot\hat{n}\,dS \qquad \text{(III.12)}$$

where $\overline{P}$ and $\overline{Q}$ are two vector functions of position and $\hat{n}$ is the outward unit normal vector. In order to apply the vector Green's theorem, we scalarly multiply the dyadic equation (III. 4) on the right by an arbitrary constant vector $\overline{a}$ and obtain a vector relation for $\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}$,

$$\nabla\times\nabla\times\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a} - k^2\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a} = \overline{a}\,\delta(\overline{r}-\overline{r}') \qquad \text{(III. 13)}$$

Let $\overline{P} = \overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}$ and $\overline{Q} = \overline{E}(\overline{r}')$, Equation (III. 12) together with Eqs. (III. 1) and (III. 13) give

$$\overline{E}(\overline{r})\cdot\overline{a} = -i\omega\mu\int_V [\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}]\cdot\overline{J}(\overline{r}')dV'$$

$$-\int_S \hat{n}'\cdot\left\{[\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}]\times\nabla'\times\overline{E}(\overline{r}') - \overline{E}(\overline{r}')\times\nabla'\times[\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}]\right\}dS' \qquad \text{(III. 14)}$$

From the Maxwell's equation $\nabla'\times\overline{E}(\overline{r}') = -i\omega\mu\overline{H}(\overline{r}')$ and the vector identities

$$\hat{n}'\cdot[\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}]\times[\nabla'\times\overline{E}(\overline{r}')] = -\hat{n}'\times[\nabla'\times\overline{E}(\overline{r}')]\cdot[\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}]$$

$$\hat{n}'\cdot\overline{E}(\overline{r}')\times\nabla\times[\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}] = \hat{n}'\times\overline{E}(\overline{r}')\cdot\nabla'\times[\overline{\overline{\Gamma}}(\overline{r},\overline{r}')\cdot\overline{a}]$$

Equation (III. 14) may be written as

$$\bar{E}(\bar{r})\cdot\bar{a} = -i\omega\mu\int_V [\bar{\bar{\Gamma}}(\bar{r},\bar{r}')\cdot\bar{a}]\cdot\bar{J}(\bar{r}')\,dV'$$

$$-\int_S \left\{ i\omega\mu[\hat{n}'\times\bar{H}(\bar{r}')]\cdot[\bar{\bar{\Gamma}}(\bar{r},\bar{r}')\cdot\bar{a}] - [\hat{n}'\times\bar{E}(\bar{r}')]\cdot\nabla'\times[\bar{\bar{\Gamma}}(\bar{r},\bar{r}')\cdot\bar{a}]\right\} dS' \qquad \text{(III. 15)}$$

Since the constant vector $\bar{a}$ may be chosen arbitrarily, Eq. (III. 15) gives

$$\bar{E}(\bar{r}) = -i\omega\mu\int_V \bar{\bar{\Gamma}}(\bar{r},\bar{r}')\cdot\bar{J}(\bar{r}')\,dV'$$

$$-\int_S \left\{ i\omega\mu[\hat{n}'\times\bar{H}(\bar{r}')]\cdot\bar{\bar{\Gamma}}(\bar{r},\bar{r}') - [\hat{n}'\times\bar{E}(\bar{r}')]\cdot\nabla'\times\bar{\bar{\Gamma}}(\bar{r},\bar{r}')\right\} dS' \qquad \text{(III. 16)}$$

The volume integral is a solution to the inhomogeneous vector Helmholtz equation with a spatially distributed source $\bar{J}(\bar{r}')$. The surface integrals are solutions to the homogeneous Helmholtz equation $\nabla\times\nabla\times\bar{E} - k^2\bar{E} = 0$ in terms of the boundary values on S. The first term in the surface integral may be interpreted as a contribution from a surface current of density $\hat{n}\times\bar{H}$ on S, and the second term may be considered as a contribution from a magnetic current sheet of density $\bar{E}\times\hat{n}$. For a perfect conductor the tangential component of $\bar{E}$ vanishes on S, then $\hat{n}\times\bar{E} = 0$.

Appendix IV

GREEN'S FUNCTION FOR THE TIME-DEPENDENT WAVE EQUATION

Consider the inhomogeneous scalar wave equation,

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -f(\bar{r}, t) \qquad \text{(IV. 1)}$$

where $f(\bar{r}, t)$ describes the distribution of sources which are functions of both position and time. The solution [39] - [40] may be written as

$$\psi(\bar{r}, t) = \int_V \int_0^\infty G(\bar{r}, t; \bar{r}', t')\, f(\bar{r}', t')\, d^3\bar{r}'\, dt' \qquad \text{(IV. 2)}$$

where the Green's function $G(\bar{r}, t; \bar{r}', t')$ satisfies the differential equation,

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = -\delta(\bar{r}-\bar{r}')\,\delta(t-t') \qquad \text{(IV. 3)}$$

To find G, we consider the Fourier representations,

$$G(\bar{r}, t; \bar{r}', t') = \int d^3\bar{k} \int d\omega\, g(\bar{k}, \omega)\, e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')}\, e^{+i\omega(t-t')} \qquad \text{(IV. 4)}$$

and

$$\delta(\bar{r}-\bar{r}')\,\delta(t-t') = \frac{1}{(2\pi)^4}\int d^3\bar{k} \int d\omega\, e^{-\bar{k}\cdot(\bar{r}-\bar{r}')}\, e^{+i\omega(t-t')} \qquad \text{(IV. 5)}$$

Substituting (IV. 4) and (IV. 5) into (IV. 3), we find

$$g(\bar{k},\omega) = \frac{1}{(2\pi)^4} \frac{1}{\left(k^2 - \frac{\omega^2}{c^2}\right)} \tag{IV. 6}$$

Hence

$$G(\bar{r}, t; \bar{r}', t') = \frac{1}{(2\pi)^4} \int d^3\bar{k} \int \frac{d\omega}{\left(k^2 - \frac{\omega^2}{c^2}\right)} e^{-i\bar{k}\cdot(\bar{r}-\bar{r}')} e^{+i\omega(t-t')} \tag{IV. 7}$$

The Green's function satisfying (IV. 3) represents the wave disturbance caused by an impulse source at $t=t'$, located at $\bar{r}=\bar{r}'$. Thus, G gives the description of the effect of this impulse as it propagates away from $\bar{r}=\bar{r}'$ in the course of time. It can then be assumed that G and $\partial G/\partial t$ should be zero for $t < t'$; i.e., no effect should precede the cause in time. Furthermore, for $t > t'$, the wave disturbance propagates outwards as a spherically diverging wave with a velocity c.

Equation (IV. 7) can be evaluated by application of the Cauchy residue theorem. However, in order to make G vanish for $t < t'$, we imagine that the poles at $\omega = \pm ck$ are displaced above the real axis by an arbitrarily small amount ϵ in the complex ω-plane. The integral over the lower half plane $(t < t')$ will then vanish, while the integral over the upper half plane will give a non-vanishing

contribution. Equation (IV. 7) may be written as

$$G(\bar{r}, t; \bar{r}', t') = \lim_{\epsilon \to 0} \frac{1}{(2\pi)^4} \int d^3\bar{k} \int d\omega \frac{e^{-i\bar{k}\cdot\bar{R} + i\omega\tau}}{k^2 - \frac{1}{c^2}(\omega - i\epsilon)^2}$$

where $\bar{R} = \bar{r} - \bar{r}'$ and $\tau = t - t'$. Then, by Cauchy's theorem, we find

$$G = \frac{c}{(2\pi)^3} \int d^3\bar{k}\, e^{-i\bar{k}\cdot\bar{R}} \frac{\sin ck\tau}{k} \qquad \text{(IV. 8)}$$

The integration over the three dimensional k-space can now be performed. We choose a coordinate system such that

$$\bar{k}\cdot\bar{R} = kR \cos \alpha$$

$$d^3\bar{k} = k^2 \sin \alpha\, d\alpha\, d\beta\, dk$$

Equation (IV. 8) becomes

$$G = \frac{c}{(2\pi)^3} \int_0^{2\pi} d\beta \int_0^{\infty} dk\, k \sin ck\tau \int_0^{\pi} e^{-ikR\cos\alpha} \sin \alpha\, d\alpha$$

$$= \frac{c}{2\pi^2 R} \int_0^{\infty} dk \sin kR \sin ck\tau$$

$$= \frac{1}{4\pi^2 R} \int_0^{\infty} \sin \frac{R}{c}\gamma \sin \tau\gamma\, d\gamma$$

$$= \frac{1}{4\pi R} \left[\delta\left(\tau - \frac{R}{c}\right) - \delta\left(\tau + \frac{R}{c}\right) \right] \qquad \text{(IV-9)}$$

where $R = |\bar{r}-\bar{r}'|$ and $\tau = t-t'$. The second term in (IV.9) does not contribute because of the conditions of realizability mentioned earlier. Therefore,

$$G(\bar{r}, t; \bar{r}', t') = \frac{1}{4\pi|\bar{r}-\bar{r}'|}\,\delta\left(t-t' - \frac{|\bar{r}-\bar{r}'|}{c}\right);\ t-t' > 0 \qquad \text{(IV.10)}$$

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13. ABSTRACT

~~This~~ report is primarily concerned with the study of electromagnetic scattering by random scatterers. Potential applications to radar detection and communication problems are stressed. In radar detection problems, it is often necessary to detect a target echo in the presence of other unwanted echoes (clutter). In order that a radar receiver can be designed to operate effectively in the presence of clutter interference, it is necessary to develop a suitable theoretical model of the clutter which can be used for the design and evaluation of detection schemes. In radio communication problems it is desirable that signals can be communicated between two non-line-of-sight points by means of electromagnetic scattering from a medium which occupies the region of space illuminated by the two antenna beams. Detailed knowledge of the scattering characteristics of the medium will enable one to select proper signal parameters for transmittal of information and optimum processing schemes.

A cloud of dipoles (chaff), dispensed in a proper region in space to act as radar reflectors, can be described as an assembly of random scatterers. When chaff dipoles are dispensed from a moving craft in space, they will in general move relative to one another. In addition, each dipole will have a tumbling motion due to effects of injection forces, body instability, and other aerodynamical properties. Since the location and motion of the individual scatterers are unknown, the scattering problems are best treated statistically.

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CONTINUATION SHEET

The bistatic radar reflecting characteristics of a randomly tumbling dipole are investigated. An expression for the scattered voltage is derived by application of the Lorentz reciprocity theorem. The correlation properties of the received signal are examined. Some statistical assumptions are made in order to obtain readily usable results.

A theoretical model is developed for the radar echo from a random collection of moving dipole scatterers. The analysis of the model takes into account some effects of scatterer rotation which have been neglected in previous work. The fluctuating characteristics of clutter echoes are also determined. The theory and some experimental results in the literature are shown to be in relatively good agreement.

The properties of random scatter communication channels are also investigated. The constitutive parameters of the scattering medium are assumed to be varying randomly with space and time. The effects of antenna gain are included in the derivation of the channel function in order to take explicit account of the fact that scatterers may flow in and out of the volume illuminated by the two antenna beams. Arbitrary polarization is assumed for both the transmitting and receiving antennas. Specific results are obtained for dipole and plasma scattering.